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PROCEEDINGS
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A METHOD OF EFFECTING ZYGOSPORE GERMINATION IN CERTAIN CHLOROPHYCEAE

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Communicated by E. W. Sinnott, June 11, 1949

Introduction.—The potential advantages of genetical studies of haplobiontic Chlorophyceae having zygotic meiosis have in many cases not been realized due to the difficulty of obtaining consistently high percentages of zygospore germination. There are comparatively few reports in the literature in which successful methods of germination are reported. Bold[1] summarized the efforts of those investigators who achieved some success by simulating in culture the environmental conditions which algal zygospores normally encounter in nature. Among these were included the transfer of zygospores to fresh media, alternate drying and wetting and other methods. Rowan,[2] using arsenic, strychnine and indole acetic acid, had considerable success with the germination of the zygospores of *Hydrodictyon*. Pocock[3] was able to germinate the zygospores of *Volvox* five or six months after their formation by placing them in dilute culture solution in direct sunlight. She concluded that direct sunlight was essential for germination, and that complete drying of the zygospores before introducing them into the culture medium was beneficial though not necessary. Lewin,[4] working with a strain of *Chlamydomonas*, has achieved up to 30% germination of the zygospores by using a method of metabolic starvation. He also concluded that complete darkness and not sunlight was essential for successful germination of the zygospores of his organism. Moewus[5] secured excellent results with *Chlamydomonas eugametos* zygospores by using a soil solution prepared without heat. Metzner[6] was able to secure only 10% germination of the oöspores of *Volvox Carteri* using Moewus' cold water extract, but was able to secure 90% germination using the synthetic medium of Uspenski and Uspenskaja.[7] Metzner was able to germinate only moist spores, although Pocock had reported previously that drying was beneficial in germinating the oöspores of another species of *Volvox*.

Experimental.—Inasmuch as the writer has obtained consistently high percentages of zygospore germination in a homothallic organism,[8] similar to *Chlorococcum*, and also with *Chlamydomonas chlamydogama* Bold, by a simple technique, the following notes as to the method used are reported herewith in the hope that they may prove effective for the zygospores of other Chlorophyceæ.

S11C-6 is a homothallic *Chlorococcum*-like alga producing gametes which fuse in pairs to form small spherical zygotes usually about 7.5 microns in diameter. After formation these zygotes grow for a period of time varying from 5 to 14 days during which they may increase up to 25 microns in diameter. After this growth period they secrete a thick, reticulate, outer wall. As the zygospores age, they seem to become filled with large oil droplets; however, they retain their green color.

TABLE 1

ZYGOSPORE GERMINATION IN RELATION TO INCUBATION AT VARIOUS TEMPERATURES

LENGTH OF EXPOSURE IN HOURS	PERCENTAGE OF GERMINATION AT INCUBATION TEMPERATURES OF:		
	37°C.	45°C.	55°C.
72	70%	..	..
60	90%	..	..
48	90%	..	..
36	30%	..	..
24	30%	..	..
20	20%	..	..
16	20%	..	..
12	0	..	..
8	0	..	..
4	0	0	0
3	0	0	0
2	0	0	0
1	0	30%	30%
1/2	0	0	0
0	0	0	0

The alga was grown in 50-ml. Erlenmeyer flasks containing 30 ml. of a modified sterile Bristol's solution[9] to which 0.03 ml. of Armour's crude liver liquid extract had been added. Although the alga will grow in the purely inorganic Bristol's solution, the growth rate is definitely increased with the addition of the liver extract. The cultures were illuminated by a fluorescent light of approximately 50 foot-candle intensity.

In preliminary attempts to obtain germination, zygospores from a month-old culture were transferred to eight small Petri dishes containing Bristol's agar with 0.15% Difco yeast extract. No germination had occurred 16 days later at which time one of the dishes was transferred to a 37°C. incubator for 48 hours. Forty-eight hours after the removal of the Petri dish from the incubator approximately 70% germination was observed in

the dish which had been incubated, but no evidence of germination was observed in the unincubated dishes. The above procedure was repeated with another of the remaining dishes and the same results were obtained. No germination of the zygospores was observed in the unincubated dishes until 35 days after their transfer to the fresh medium when an occasional one could be found germinating.

In order to verify these results, a series of Petri dish cultures was prepared using the zygospores from the same culture of S11C-6, now however, approximately 60 days old. Zygospores were again transferred to Bristol's yeast agar in small Petri dishes and incubated for various periods. The results are summarized in table 1. Percentages are approximate and represent estimates based on microscopic examination.

Maximum germination of the 60-day old zygospores occurred within 48–72 hours after exposure of the zygospores to a temperature of 37°C. for 48–60 hours.

Results comparable to the best previously achieved were also obtained using zygospores from a 15-day old culture. Zygospores from younger cultures are difficult to secure in quantity, inasmuch as they do not appear in abundance with their characteristic thick walls until several weeks after the culture has been started, and until these walls appear, it is difficult to distinguish the zygospores from vegetative cells.

Experiments have also been conducted to determine if heat had a similar effect on the zygospores of *Chlamydomonas chalmydogama* Bold. The cultures were grown in a variety of liquid media including Knop's and Volvox solutions. Zygospores were obtained by mixing the heterothallic strains of the alga in hanging drops. Zygospores of various ages were transferred to Petri dishes containing Bristol's agar and incubated at 37° C. for 48 hours. Approximately 90% germination was observed in all cultures within 48–96 hours after the heat treatment.

Discussion.—No attempt is made here to speculate as to the relation of the heat treatment to the physiology of the zygospores or as to its rôle in germination. This stimulatory effect of high temperature suggests the work of Shear and Dodge,[10] who have shown that the ascospores of *Neurospora tetrasperma* can be induced to germinate by a short period of heating. Goddard,[11] working with *Neurospora*, found evidence pointing to the presence of three phases in the rate of respiration of ascospores, corresponding to the stages of dormancy, activation and germination. Goddard and Smith,[12] investigating further the respiration of the ascospores of *Neurospora*, obtained results showing that the marked increase in the respiratory rate and then subsequent germination of the heat-treated ascospores was probably due to the activation of carboxylase by the heat.

Notwithstanding the fact that these investigations of the relation between high temperature and zygospore germination are perhaps not ex-

tensive enough to make many conclusive statements as to the exact conditions under which the zygospores of S11C-6 and *Chlamydomonas chlamydogama* Bold will germinate, the writer is of the opinion that the method may form a useful point of departure for other investigators who may be confronted with the problem of zygospore germination. It is hoped that this method of inducing germination may prove especially useful to those concerned with genetical problems in which it would be decidedly advantageous to effect zygospore germination in the shortest possible time and with the greatest possible frequency.

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[2] Rowan, M., Unpublished Thesis, Columbia Univ. (1937).
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[5] Moewus, F., *Zeit. Ind. Abst. Ver.*, **78,** 418 (1940).
[6] Metzner, J., *Bull. Torrey Club*, **72,** 86 (1945).
[7] Uspenski, E. E., and Uspenskaja, W. J., *Zeits. Bot.*, **17,** 273 (1925).
[8] This organism will be referred to in this report as "S11C-6," inasmuch as no final conclusion as to its identity has yet been reached. It was isolated from the soil of San Marcellino, Luzon, Philippine Islands, by Dr. Harold C. Bold of Vanderbilt University.
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[10] Shear, C. L., and Dodge, B. O., *J. Agr. Res.*, **54,** 1019 (1927).
[11] Goddard, D. R., *J. Gen. Physiol.*, **19,** 45 (1935).
[12] Goddard, D. R., and Smith, P. E., *Plant Physiol.*, **13,** 241 (1938).

*STUDIES ON THE BIOCHEMICAL GENETICS OF YEAST**

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Communicated June 21, 1949

The mechanism of inheritance in *Saccharomyces cerevisiae* Hansen has been a subject of controversy, for it has not been clear whether, in this yeast, the characteristic mode of inheritance is Mendelian or non-Mendelian.

The work of Lindegren and Lindegren,[1] and Spiegelman,[2] on the inheritance of adaptive enzyme systems, has pointed to a mechanism of inheritance mediated by cytoplasmic, self-duplicating units. Lindegren and Lindegren[3] have recently presented evidence which was interpreted as a gene-to-gene transfer of components, and for which the concept of a "depletion mutation" was proposed. In the face of hypotheses of such an unusual nature, it becomes worth while to examine the possibility of devising critical experiments to determine what the basic mechanism of inheritance is in this organism.

It is of prime importance in such studies that characters controlled by single genes be studied as criteria by which to judge the genetic process. The technique first used by Beadle and Tatum[4] in *Neurospora*, viz., the induction of biochemically deficient mutants by irradiation, seemed well-suited to this purpose. That such mutants can be produced in yeast has been shown by Reaume and Tatum,[5] who recovered mutants after treatment with nitrogen mustard. It has been amply demonstrated in other organisms, such as *Neurospora*,[6] *Ophiostoma*,[7] and *Escherichia coli*,[8] that the mutant characters, i.e., the biochemical deficiencies, are inherited as though controlled by single genes. Accordingly, haploid strains of *S. cerevisiae* (very generously supplied by Dr. C. C. Lindegren) were treated with ultra-violet irradiation, and mutants were recovered by an appropriately modified screening[9] and detection[10] technique. The mutants that were obtained are listed in table 1.[11] They may be characterized, in general, as being unable to grow in the absence of the required metabolites for at least 72 hours at 30°C.

TABLE 1

BIOCHEMICALLY DEFICIENT MUTANTS OF *S. cerevisiae* PRODUCED BY ULTRA-VIOLET IRRADIATION OF HAPLOID STRAINS

CULTURE IRRADIATED	NUMBER AND KINDS OF MUTANTS
*a**	1 yeast nucleic acid
	2 methionine
	1 tryptophan
	1 adenine
	1 methionine, adenine, histidine
*α**	1 unknown factor
Tryptophanless *a*†	1 adenine
	1 uracil
	3 methionine
	3 pantothenicless
Methionineless *α*‡	3 adenine
	1 uracil

* Cultures kindly supplied by Dr. C. C. Lindegren.

† Tryptophanless mutant from irradiation of *a*.

‡ Mutant obtained following mustard gas treatment[5] and kindly supplied by Mr. S. E. Reaume, who also gave us a pink, adenineless *α* haploid.

Lindegren and Lindegren[12] were the first to report a heterothallic condition in *S. cerevisiae*. The symbols "*a*" and "*α*" were given to the major allelic gene pair controlling mating type. These authors[13] devised a mass crossing procedure based on the stability of the heterothallic condition. Winge and Roberts[14] have criticized this procedure on the grounds that aberrant results may be obtained due to selfing. To obviate their objections, we adopted the following procedure: complementarily deficient mutants of opposite mating type were crossed as suggested by Lindegren

and Lindegren,[13] and after incubation for 72 hours at 25°C., the cell suspension was plated on agar containing none of the factors that the deficient haploids required. By this means, one recovers only heterozygous, "prototrophic,"[15] diploids, which resemble the wild-type diploid in nutritional requirements. Using mutants deficient for 2 to 3 metabolites, it was possible to rule out statistically the probability of recovering a reverted haploid clone. Since it was possible that a "prototrophic"

TABLE 2

SEGREGATION OF CHARACTERS IN ASCOSPORES FROM THE CROSS a $Th^-Ur_1^-Ur_2^+Me_1^+ \times$ α $Th^+Ur_1^+Ur_2^-Me_1^-$

ASCOSPORE	MT	Th	Me_1	Ur	Ur_1, Ur_2 DISTRIBUTION
A61.1	α	−	+	−	$Ur_1^-Ur_2^+$
A61.2	α	+	+	−	$Ur_1^+Ur_2^-$
A61.3	a	−	−	−	$Ur_1^-Ur_2^-$
A61.4	a	+	−	+	$Ur_1^+Ur_2^+$
A67.1	a	+	−	−	$Ur_1^-Ur_2^+$
A67.2	α	−	−	−	$Ur_1^+Ur_2^-$
A67.3	a	+	+	−	$Ur_1^-Ur_2^-$
A67.4	α	−	+	+	$Ur_1^+Ur_2^+$
A68.1	a	−	−	−	$Ur_1^+Ur_2^-$
A68.2	α	+	−	−	$Ur_1^+Ur_2^-$
A68.3	α	+	+	−	$Ur_1^-Ur_2^+$
A68.4	a	−	+	−	$Ur_1^-Ur_2^+$
A69.1	α	−	−	+	$Ur_1^+Ur_2^+$
A69.2	a	+	+	−	$Ur_1^-Ur_2^-$
A69.3	α	+	−	+	$Ur_1^+Ur_2^+$
A69.4	a	−	+	−	$Ur_1^-Ur_2^-$
A70.1	α	−	+	−	$Ur_1^+Ur_2^-$
A70.2	a	+	−	−	$Ur_1^+Ur_2^-$
A70.3	a	−	−	−	$Ur_1^-Ur_2^+$
A70.4	α	+	+	−	$Ur_1^-Ur_2^+$

The numbering system used for identification of ascospores is as follows: the letter "A" prefixes the ascus number; a "period" follows the ascus number, and after the "period" comes the ascospore number. Thus A61.1, A61.2, A61.3 and A61.4 refer to the four ascospores of ascus 61. The ascospore numbers are only identification numbers, and have no significance with regard to the sequence of isolation.

colony might represent a synergistic association of haploids and diploids, a population analysis was carried out on a prototroph isolate from the cross a $Th^-Tr^-Ur_1^-Me_1^+ \times \alpha$ $Th^+Tr^+Ur_1^+Me_1^-$.‡ The population, was homogeneously independent as regards nutritional requirements indicating that the single colony isolate represents a pure diploid clone. The final proof was the demonstration of heterozygosity in the segregating ascospores. This over-all mating procedure preserves the advantages of the mass crossing technique, and insures that the material to be examined

genetically has arisen from heterothallic conjugation. This would render unnecessary the precautions of Winge and Laustsen,[16] who carried out crosses of single spores in hanging drop preparations under continual microscopic observation.

Sporulation of prototrophic diploids was induced by the Lindegren and Lindegren[17] modification of the gypsum slant[18] method, and by the procedure of Stantial.[19] The asci analyzed, in all cases here reported, were taken from gypsum slants. The four ascospores of individual asci were separated by microdissection using a modified Fitz micromanipulator.[20] They were then allowed to germinate at 30°C. in hanging drops. The medium (Lindegren, Personal Communication) used was as follows: liquid yeast extract (Anheuser Busch No. 3), 6 ml.; peptone, 5 g.; $MgSO_4 \cdot 7H_2O$, 1 g.; KH_2PO_4, 2 g.; sodium lactate (50%), 7.2 g.; dextrose, 40 g.; and water to make 1000 ml. After incubation for 24 hours in hanging drop cultures, the individual haploid clones were transferred, for purposes of further multiplication, to tubes containing nutrient broth, and from these to agar slants, to produce stock cultures.

TABLE 3

RELATION BETWEEN GENOTYPE AND PHENOTYPE IN SEGREGATIONS INVOLVING TWO NON-ALLELIC GENES WITH THE SAME APPARENT PHENOTYPIC EXPRESSION (URACIL REQUIREMENT)

PHENOTYPIC SEGREGATION (INDEPENDENT:DEPENDENT)	GENOTYPES
0:4	$Ur_1^-Ur_2^+$, $Ur_1^-Ur_2^+$, $Ur_1^+Ur_2^-$, $Ur_1^+Ur_2^-$
1:3	$Ur_1^+Ur_2^+$, $Ur_1^-Ur_2^-$, $Ur_1^+Ur_2^-$, $Ur_1^-Ur_2^+$
2:2	$Ur_1^+Ur_2^+$, $Ur_1^+Ur_2^+$, $Ur_1^-Ur_2^-$, $Ur_1^-Ur_2^-$

Using this over-all procedure, analyses of several crosses have been carried out. In those cases where only three out of four spores survived, the practice has been followed, for linkage calculations, of assigning the genetic character of the fourth spore by "difference," based on a 2:2 segregation for each single character. This seemed justified in view of the regular results obtained when all four spores survived. In some crosses, the characters of "disperse" and "non-disperse" have been included in the analysis. "Disperse" (D^+) refers to homogeneous growth in liquid shake cultures with dextrose as a carbon source in a synthetic medium,[21] and "non-disperse" (D^-) signifies flocculence under the same conditions. "Disperse" appears to be dominant over "non-disperse," since the prototrophic diploid arising from the cross "disperse" × "non-disperse" is generally "disperse." However, with sucrose or fructose serving as a carbon source, the normally "disperse" diploid grew flocculently, indicating that caution must be exercised in evaluating this character. Pending further study, this character is included in the analyses only tentatively.

To determine whether phenotypically similar mutants were genetically allelic, the following procedure was used: two mutants of opposite mating type, deficient for the same metabolite, were mixed in broth to permit copulation[13] and incubated at 25°C. for 72 hours. The cells were then centrifuged, washed and plated on synthetic agar[21] lacking the metabolite for which the common requirement existed. If growth of the diploid cells occurred in 48 hours at 30°C., the mutants were considered as being non-allelic. If there were no growth, then they were taken to be allelic. The test was confirmed genetically in an analysis of two uracilless mutants (Ur_1^- and Ur_2^-). The results are shown in table 2. The fact that uracil-independent ascospores are segregated out proves that the genes are non-allelic. The analysis in the last column in table 2 is based on the theoretical genetic segregations shown in table 3.

By back-crossing the haploid segregants to the two parental uracilless types, and carrying the diploids through a prototroph-recovery procedure, it was possible to assign the appropriate genetic constitution in those cases where it was not readily apparent by inspection. The reasoning behind the genotype-phenotype relation in table 3 is as follows: if Ur_1 and Ur_2 are two separate genes concerned in the biosynthesis of uracil, and falling either in a single biosynthetic chain of reactions, or in two separate series neither of which alone can support growth, then assuming the non-function of either gene in a haploid, it should be unable to grow without an appropriate exogenous metabolite. The genetic tests which were made with the marked strains confirmed the basic reasoning, in that heterozygosity could be demonstrated where theory demanded it should exist. It should also be noted in table 2 that mating type, and the requirements for thiamin (Th) and methionine (Me_1) segregated regularly.

The two uracilless mutants were tested by the auxanogram technique for their responses to various compounds. Both Ur_1^- and Ur_2^- were found to be capable of using uridine, but neither could use cytidilic or guanylic acids. However, Ur_1^- responded to thymine, while Ur_2^- did not. Loring and Pierce[21] have reported a pyrimidine mutant of *Neurospora* which was able to utilize either thymine or uracil, but which could also utilize cytidilic acid. Both of our uracilless yeast mutants appear to differ from these *Neurospora* mutants.

The prototroph-allelism test was also applied to the methionineless and adenineless mutants. At least two classes of methionineless and 5 classes of adenineless mutants were uncovered. The methionineless mutants are unable to utilize the *D*-isomer, in contrast to reported results in *E. coli*.[22] The methionineless yeasts are able to use cysteine partially, but cannot use cystine (since in the original screening procedure, cystine was present but could not replace the methionine requirement). All of the adenineless mutants can use hypoxanthine as well as adenine. They grow to varying

TABLE 4

SEGREGATION OF CHARACTERS IN THE ASCOSPORES FROM THE CROSS *a* D$^+$Th$^-$Tr$^-$Ur$_1^-$Me$_1^+$ × *α* D$^-$Th$^+$Tr$^+$Ur$_1^+$Me$_1^-$

SPORES → ASCUS	GENETIC CONSTITUTION OF THE FOUR ASCOSPORES																					
	1						2						3						4					
	MT	D	Th	Tr	Ur$_1$	Me$_1$	MT	D	Th	Tr	Ur$_1$	Me$_1$	MT	D	Th	Tr	Ur$_1$	Me$_1$	MT	D	Th	Tr	Ur$_1$	Me
A5	*α*	−	−	+	−	−	*a*	−	+	−	−	−	*a*	+	+	−	+	+						
A7	*a*	−	−	+	+	−	*α*	−	−	−	−	+	*α*	+	+	−	+	+						
A18	*α*	−	+	−	+	+	*a*	+	−	−	−	−	*a*	+	−	+	+	+						
A19	*α*	−	+	−	+	−	*α*	+	−	+	−	−	*a*	+	−	−	−	+	*a*	−	+	+	+	+
A23	*α*	−	+	+	−	+	*a*	−	+	−	−	−	*α*	+	−	−	+	+	*a*	+	−	+	+	−
A24	*α*	−	−	+	−	+	*a*	−	+	−	−	−	*α*	+	−	+	+	−	*a*	+	+	−	+	+
A25	*a*	−	−	−	+	−	*α*	+	+	+	−	−	*α*	+	+	−	+	+	*a*	−	−	+	−	+
A26	*a*	+	+	+	+	−	*a*	−	+	+	+	−	*α*	−	−	−	−	+	*α*	+	−	−	−	+
A27	*a*	+	−	+	−	+	*α*	−	+	−	+	+	*α*	−	−	−	−	−	*a*	+	+	+	+	−
A28	*α*	+	−	+	+	−	*α*	+	+	−	−	−	*a*	−	+	+	+	+						
A29	*a*	−	+	−	−	+	*α*	−	−	−	−	−	*α*	+	−	+	+	−						
A32	*a*	+	+	+	−	−	*α*	−	−	−	−	+?	*α*	−	−	−	+	+	*a*	+	+	+	+	+
A33	*a*	−	−	+	−	−	*α*	+	−	+	−	+	*α*	−	+	−	+	+						
A34	*a*	+	−	−	+	−	*α*	−	+	+	−	−	*a*	−	+	+	+	+	*α*	+	−	−	−	+
A35	*a*	+	−	−	+	−	*α*	−	+	+	+	+	*α*	+	−	+	−	−						
A37	*α*	−	−	+	−	−	*α*	−	+	−	−	+	*a*	+	+	−	+	+	*a*	+	−	+	+	−
A38	*a*	−	−	+	−	−	*a*	−	−	+	−	+	*α*	+	+	−	+	+	*α*	+	+	−	+	−
A39	*α*	+	−	+	−	+	*a*	+	+	−	+	+	*α*	−	−	−	+	−	*a*	−	+	+	−	−
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A44	*a*	−	−	+	+	−	*α*	+	+	+	−	+	*α*	+	+	−	−	−	*a*	−	−	−	+	+
A45	*α*	+	−	+	+	−	*α*	+	+	−	−	−	*a*	−	−	−	+	+	*a*	−	+	+	−	+
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A47	*α*	−	+	−	+	−	*α*	+	−	−	+	+	*a*	−	+	+	−	−	*a*	+	−	+	−	+
A48	*a*	+	−	−	+	−	*α*	+	−	+	−	+	*a*	−	+	+	+	−	*α*	−	+	−	−	+

extents on adenosine, but only one can use adenylic acid and adenosinetriphosphate (α, adenineless, pink strain of Reaume and Tatum[5]). Work is in progress to characterize these mutants further.

To illustrate the genetic results obtained, the analysis of the cross a $D^+Th^-Tr^-Ur_1^-Me_1^+ \times \alpha$ $D^-Th^+Tr^+Ur_1^+Me_1^-$ is presented in table 4. Other characteristics included in this cross were growth rate and drug resistance, but these do not appear to segregate as single factors, and will be discussed elsewhere. As is readily apparent from the data in table 4, the clones derived from single ascospores showed considerable regularity (there being only one questionable result). Mendelian principles of segregation and independent assortment certainly appear applicable to this material. X^2 (Chi-square) analysis was carried out for all possible combinations of genes, and there was evidence of linkage between two of the characters, D and Ur_1 ($X^2 = 7.2$, one degree of freedom, $p = 0.007$). The range of X^2 values for the other characters was from $X^2 = 0$ (MT:D, D:Tr, and Th:Me_1) to $X^2 = 3.7$ (MT:Tr). Of outstanding significance is the complete confirmation of Lindegren's observation[12] of two stable mating types in this strain of *S. cerevisiae*. Although the strength of the mating reaction varies with each clone, as reflected by the frequency of appearance of conjugating cells in microscopic examination of crosses of segregants $\times$ tester stocks, the heterothallic status of this strain seems clear. These results, and those obtained with other crosses, indicate that, for the characters studied, inheritance is Mendelian and completely analogous to that in higher organisms.

Discussion.—The problem posed at the outset of this work was to devise means whereby the inheritance of characters controlled by single genes could be studied under controlled conditions. The stability of the haploid mating types[12] allows considerable latitude in the manipulations to which they can be subjected, and still retain their mating ability. Thus, it was relatively simple to produce stable mutants which possessed the power to conjugate with mutants of the opposite mating type. The prototroph-recovery procedure insures the isolation of diploids arising from heterothallic conjugation. In addition, it is a very effective procedure in recovering diploids where conjugation is poor. Thus, for example, if only 2 out of 10^6 cells were able to conjugate, the diploid heterozygote could readily be recovered from a heavily seeded plate of minimal agar. The dispatch and sureness with which this procedure also screens for allelism is of great potential importance in studies of biochemical genetics with this organism. The ability of *S. cerevisiae* to survive either as a haploid or a diploid makes it a very flexible tool for different types of genetic investigations.

It seems reasonable to conclude that the characters studied in these crosses segregated according to Mendelian principles. One cannot state

that all of the genes of this yeast segregate regularly, although it seems not unlikely that the same basic pattern should be found in most cases where single genes are being studied. With regard to the inheritance of adaptive enzymes, little can be said. In order to decide whether the mechanism of inheritance is through cytoplasmic genetic units as suggested by Lindegren and Lindegren[1] and Spiegelman,[2] or by genes controlling rapid and slow fermentation, as reported by Winge and Roberts,[14] requires further experimental evidence. It would appear worth while to determine how many processes are actually being measured in the course of adaptation, and then to try to follow the inheritance of a single reaction instead of an entire complex, for the purpose of clarifying the inheritance of this type of reaction.

Summary.—Biochemically deficient mutants of *Saccharomyces cerevisiae* Hansen were obtained after ultra-violet irradiation of haploid mating strains. Various mutants were crossed, and heterozygous diploids obtained by a prototroph-recovery procedure. The same procedure was used in studying allelism between phenotypically similar mutants. Genetic analyses indicated that the characters studied segregated independently and regularly according to Mendelian principles.

* This material was abstracted from a dissertation, presented by the senior author for the degree of Doctor of Philosophy in Yale University.

† This work was done while the senior author was the recipient of a Standard Brands, Inc., Fellowship in Microbiology at Yale.

‡ The symbols used to represent genetic characters are as follows: a and α = mating type (MT); Ur = uracil; Me = methionine; and Ad = adenine. Superscripts $^+$ and $^-$ refer to nutritional status ($^+$ = independent, $^-$ = dependent), and independence $^{(+)}$ is dominant over dependence $^{(-)}$. Subscripts refer to different alleles with the same apparent phenotypic expression. Thus, $a\ Th^-Tr^-Ur_1^-Me_1^+ \times \alpha\ Th^+Tr^+Ur^{1+}Me_1^-$ refers to a cross between a thiaminless, tryptophanless, uracilless (locus 1), and α methionineless (locus 1). + and − in the body of a table refer to genetic constitution, i.e., independence and dependence, respectively.

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VARIATION IN THE AMOUNT OF DESOXYRIBOSE NUCLEIC ACID IN DIFFERENT TISSUES OF TRADESCANTIA

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It has become a text book generalization that all cells of an organism have the same chromosomal constitution. To the cytogeneticist such a generalization offers little hope for a solution of the great problem of differentiation, though the outlook is not quite so discouraging on the chemical level. The biochemical analyses of Chargaff and others[3, 13] have established that the composition of desoxyribose nucleic acid (DNA), which forms so important a constituent of the chromosome, is not always the same and may thus furnish a basis for variation. Further, it has recently been shown (Pollister and Leuchtenberger,[9] and Leuchtenberger[7]) that physical differences resting on various degrees of polymerization may underlie further variations of DNA, and therewith some very important steps in the analysis of differentiation have been initiated.

Indeed, the cytological investigations of Geitler and a few other workers have furnished evidence that the initial generalization was altogether too sweeping in its claims. In many organisms there are tissues whose cells have higher numbers of chromosomes than are encountered in the diploid complement. The recent work of Geitler and of Huskins has made it clear that polyploidy or polysomaty is characteristic of many fully differentiated tissues, and, as Huskins points out, such "data on endopolyploidy permit us to assume quantitatively different gene action in different tissues" (Huskins).[5]

But, though the cytologist was prone to accept an identity of chromosomal number for all normal cells in an organism prior to these discoveries, he could not possibly escape the fact that the chromosomes in the cleaving

egg are often larger than those of the blastula and gastrula, and that in later stages the chromosomes of some tissues are uniformly larger than those of others. Here is a quantitative difference which, oddly enough, was rarely considered. Possibly this was due to a recognition of the possibility that in such instances the larger chromosomes have their materials in a more diluted condition than the small ones, and that their essential constituents therefore did not differ in amount. To be sure, visual comparison under the microscope seems, in at least some cases, to speak against such an explanation, but it may well be recognized that there is involved such a multiplicity of factors in the optical and staining properties of chromosomes that a final decision would be very difficult to obtain.

The recent findings of Boivin, Vendrely and Vendrely,[2] Vendrely and Vendrely[12] and Mirsky and Ris,[8] would seem to suggest that such caution is indeed warranted. Working on tissues in vertebrates, they arrived at the conclusion that all the diploid cells of an organism carry identical amounts of DNA. Their conclusions were based on a gross biochemical analysis of nuclei separated from the cytoplasmic constituents of the cell. In evaluating these results it should be noted that the amount of DNA determined for the nuclei of certain tissues by the Vendrelys differs from the amount reported for the same tissues by Mirsky and Ris. Indeed, in their values for bull sperms the foregoing authors differ not only from each other but also from Zittle and O'Dell.[14] Any attempt to generalize on these findings must therefore await the results of further analyses.

The question at issue is so important that an investigation of completely different material seemed warranted. For this purpose the plant Tradescantia paludosa was used, all material coming from plants of the same clone.[1] The tissues tested were root tips, the petals of small, unopened buds, the basal parts of leaves and certain meiotic stages. All were fixed in Carnoy's acetic alcohol (1:3) and sectioned at 15 μ. Staining was always done in exactly the same way under standardized conditions and to further assure an identity of treatment, sections of root and bud tissue and, in other cases, bud and leaf tissue were mounted on the same slides. Both the root and bud tissues studied showed only metaphases with the diploid number of 12 chromosomes; in the leaf tissue almost no mitoses were encountered.

The Feulgen reaction, which is known to have a high specificity for DNA (Stowell,[11] DiStefano[4]), was used throughout. Its reproducibility has recently been shown by Leuchtenberger in animal material. In order to measure the amounts of Feulgen dye in individual nuclei, we used the photometric, microscopic method of Pollister and Ris[10] which permits the estimation of relative amounts of colored precipitates in the nuclei of fixed and stained sections. The Feulgen reaction was measured in terms of the absorption of the green light isolated from a zirconium lamp by a Farrand interference filter with a peak transmission in parallel light at 557 millimi-

crons. The photo tube was an R. C. A. Photomultiplier, Type 931 A. The mean extinctions (mean of 10–20 nuclei) obtained appear in the Table as E_{550}. For a better evaluation of the absorption data, extinction values were considered in terms of nuclear sizes and this was done by using $E_{550} \times \pi r^2$ as an arbitrary unit (see Column 6 in table 1). For the absorption measurements of the Feulgen dye in root, bud and leaf tissues, nuclei of the same mean nuclear diameter (8.6 microns) were selected; similarly, comparisons between prophases in root tips and meiotic pachytene stages were made in nuclei with a mean diameter of 10.8 microns. The DNA per nucleus was computed as described by DiStefano and, since such computations involve several assumptions and are subject to some errors (Leuchtenberger[7]), it must be considered that these DNA values are relative and not absolute.

TABLE 1

COMPARISON OF DESOXYRIBOSENUCLEIC ACID (FEULGEN) IN DIFFERENT TISSUES OF TRADESCANTIA PALUDOSA

STAGE	TYPE OF TISSUE	MEAN NUCLEAR DIAMETER IN MICRONS	NUMBER OF NUCLEI MEASURED	E_{550}	$E_{550} \times \pi r^2$	RELATIVE AMOUNT OF DNA IN 10^{-6} MG.* PER NUCLEUS
Interphase	Root tip	8.6	20	0.443 ± 0.014	25.7 ± 0.78	5.5
Early prophase	Root tip	10.8	10	0.570 ± 0.025	51.4 ± 2.40	11.0
Interphase	Leaf	8.6	17	0.735 ± 0.02	42.7 ± 1.29	9.0
Interphase	Bud tapetum	8.6	20	0.969 ± 0.02	56.3 ± 1.68	12.0
Meiosis pachytene	Bud anther	10.8	10	0.939 ± 0.02	84.7 ± 2.4	18.0
Meiosis microspore interphase	Bud anther	7.0	10	0.733 ± 0.08	25.7 ± 2.9	6.0

* Values in these columns involve several assumptions and therefore are only relative.

On the basis of measurements such as are tabulated here, it appears that interphase nuclei of different somatic tissues of Tradescantia (root, bud and leaf) do not carry identical amounts of DNA even though, in bud and root tissue at least, the same (diploid) chromosome number is involved. So large are the differences involved that there would seem to be no escape from this conclusion.

It should further be noted that pachytene nuclei of meiotic cells show considerably more DNA than equally sized somatic prophase nuclei. Also, such pachytene nuclei carry three times as much DNA as the interphase nuclei after the second division (microspores). It is likely that the actual ratio is the theoretically expected 4:1 and that the values obtained for the pachytene nuclei appear low because of the uneven distribution of chromosome material at that stage. Especially striking is the fact that the microspore carries an amount of DNA corresponding very closely to that of the root tip nucleus. This is a coincidence which in itself demonstrates that haploid nuclei resulting from two meiotic divisions do not necessarily carry only half the DNA contained in diploid, somatic cells.

It is of interest that if Tradescantia tissues are fixed in 20% Formalin, absorption measurements of Fuelgen-stained nuclei give extinction values about twice as great as after Carnoy fixation. In spite of this increase the ratios of relative amounts of DNA in different tissues (for instance, between bud and root) are essentially the same for both types of fixation.

The unavoidable conclusion is that the amount of DNA carried in a given chromosome may vary in different tissues. The explanation of such variation may involve one of two basic conceptions. Thus, it is possible that the DNA has no exact quantitative relation to the basic structure of the chromosome and may vary according to the age or other physiological conditions of the nucleus or its surroundings. An interpretation of this sort has been made by several cytologists and is implied in such terms as "nucleic acid starvation" and "supercharging of nucleic acid." As long as the structural and chemical composition of the chromosome is not better known, such a possibility must be considered. Another interpretation rests on the cytological concept that each chromosome is composed of a number of threads or chromonemata. If each of such threads is associated with a definite and identical quantity of DNA, then the total amount of DNA in a chromosome depends on the number of chromonemata of which it is composed. A chromosome with a large number of chromonemata—that is, of a high degree of polyteny—would therefore contain more DNA than an otherwise homologous chromosome of a lower degree of polyteny.

On the whole, this second interpretation is at present the most acceptable, if only because we already know definitely that polyteny exists. On such a basis the haploid chromosome complement of the Tradescantia microspore carries practically as much DNA as the diploid root tip because each of its chromosomes is composed of twice as many chromonemata. It is a strong argument for the claim that reorientation in some of our cytogenetic concepts is called for—especially with respect to the numerical relation of the chromonemata to the chromosomes.

The fact that the different amounts of DNA in Tradescantia do not show any obvious multiple relation to each other may be due to several causes. First of all is the possibility that, for mechanical reasons, the photometric determinations are still subject to a certain degree of error, as we have already stated. But it is also conceivable that, as Huskins and Steinitz[6] have pointed out in their analysis of various types of endomitosis in Rhoeo, certain chromosomes may divide prior to others of the same complement. This may also obtain in the internal replication involved in polyteny, with the result that intermediate values are obtained for some tissues.

If DNA has the exact quantitative relation to the individual chromonema that has been considered in the foregoing, the recent conclusions of Boivin and the Vendrelys are not surprising. They mean only that the vertebrate tissues analyzed all have nuclei in which the same degree of poly-

teny obtains. Such data do not necessarily give final information on the number of chromosomes in such tissues for, as our findings in Tradescantia show, the quantity of DNA contained in a nucleus is probably dependent on the grade of polyteny in its chromosomes.

Summary.—Contrary to the findings of several workers for the tissues of vertebrates, the nuclei in different tissues of the plant Tradescantia carry different amounts of DNA. This is not attributable to various degrees of polyploidy in this case but may be due to different degrees of polyteny that obtain in the various tissues.

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*CHROMOSOMES OF A FOX HYBRID (Alopex-Vulpes)**

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Numerous attempts have been made to obtain offspring by mating animals classified as belonging to different species or even different genera. When young have been the result of such matings, they often prove to be semisterile or completely sterile in matings *inter se* or back to either of the parental groups. The mule has long been regarded as the classical example of an intergeneric hybrid which falls into this category. From time to time since the red fox (*Vulpes vulpes* L.-*V. fulva* species) and the arctic fox (*Alopex lagopus* L.) have been brought under domestication, breeders have produced hybrids from the two species with the idea that, by back-crossing, the size of the arctics could be increased and their higher productivity introduced into the color phases of the red fox. This objective has not met

with success to date, however, apparently as a result of the complete sterility of this intergeneric hybrid.

Cole and Shackelford[1] have described *Alopex-Vulpes* hybrids and discussed their breeding behavior in detail. One of the males from the group designated as Case 8 by these workers furnished material for the chromosome studies reported in this paper. Tissues from the black (silver) color phase of the red fox came from the University of Wisconsin ranch and those from the arctic fox were collected through the courtesy of Messrs. John A. Fromm and Loyal Wells of Fromm Bros., Nieman and Co., Inc., Thiensville, Wis.

Individuals of the two parental species and the hybrid were castrated during the breeding season (1945) in order to obtain meiotic as well as mitotic divisions. Although the breeding seasons of the two species overlap, that of the arctic fox averages a later date than that of the red fox in this locality. Consequently, the testes from the arctic fox were taken March 21, and those from the red fox February 10. It was assumed that the height of the breeding season for the hybrid was probably between those for the parent species, so one testis was removed March 7, 1945. Studies of this material suggested that the testis of the hybrid might have been removed too early in the breeding season, so the next year (1946) the remaining testis was taken at approximately the date found satisfactory for the arctic fox, namely March 19.

Seminiferous tubules were teased from the testes directly into Carnoy's alcohol-acetic acid-chloroform fixative (7:2:1). An hour later the material was transferred to 80% alcohol for storage. Temporary aceto-carmine smears were made and the desired division figures observed.

Andres[2] and Wipf and Shackelford[3] reported 34 as the diploid chromosome number for *Vulpes*. The material of the red fox used in this study came from a different animal from any of those used in the 1942 report,[3] and confirmed 34 as the chromosome number for *V. vulpes* (figure 1 *A*). The present study is also in agreement with Andres'[2] count of 52 for *Alopex* (figure 3 *A*). Wodsedalek[4] gave 42 as the number for the red fox, Bishop[5] 32 and Makino[6] 38.

Considerable differences can be noted in size and shape between the chromosomes of the two parent species at mitosis; the chromosomes of the red fox are much larger and longer, the largest being more than twice the length of the largest in the arctic. Both species show satellite attachments in one pair. No sex chromosomes can be distinguished on the basis of heteropycnosis, differences in size or other morphological characters. Within each species it is relatively easy to pair a chromosome with another which appears to be its homologue (figures 1 *B* and 3 *B*). At the first reduction division the chromosomes of *Vulpes* (figure 4) are larger than

FIGURE 1 (*A*)

Vulpes chromosomes on a mitotic equatorial plate. $2n = 34$.

FIGURE 1 (*B*)

Chromosomes from figure 1 (*A*) arranged as homologous pairs.

FIGURE 2 (*A*)

Mitotic equatorial plate from an *Alopex-Vulpes* hybrid showing a chromosome number of 43.

Captions continued on facing page.

those of *Alopex* (figure 6) as was the case in mitotic divisions, and chromosomes that appear morphologically similar are paired in both species.

Polar views of the equatorial plate stage of mitotic figures from the *Alopex-Vulpes* hybrid show 43 chromosomes (figure 2 *A*). Some of the chromosomes are large and can readily be recognized as coming from the *Vulpes* parent; others are smaller and more closely resemble those of *Alopex* (figure 2 *B*). At the first reduction division a mixture of paired and single chromosomes is found on the equatorial plate (figure 5). Side views of equatorial plates, anaphases and telophases show no abnormalities such as precocity or lagging in either mitotic or meiotic divisions.

It is doubtful that mature sperms are formed in the hybrid. The cell type which comes nearest resembling a sperm is large and irregular in outline and lacks the compact appearance of *Alopex* and *Vulpes* sperm heads. Whatever may be the causative element, it is possible that the sterility of the *Alopex-Vulpes* hybrid, at least in the male, stems from the inability to produce mature gametes.

Other workers have reported irregularities of various kinds in the formation of gametes in a number of interspecific crosses in mammals. Wodsedalek[7] noted that in the mule, chromosomes were often scattered on the spindle as well as in the cytoplasm at the metaphase stage and multiple and distorted spindles were often seen. Dice[8], working on dog-coyote hybrids, reported that "the maturation divisions seem to be highly abnormal." He found no divisions at any stage later than early anaphase of the first reduction division. In cattle-bison hybrids (Deakin, Muir and Smith[9]), the seminiferous tubules were found to be in a degenerative state with no germinal epithelium. There were indications of active mitosis and a degree of differentiation of spermatids toward spermatozoa in some of the tubules.

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FIGURE 2 (*B*)

Chromosomes from figure 2 (*A*) placed in a linear arrangement.

FIGURE 3 (*A*)

Alopex chromosomes on a mitotic equatorial plate. $2n = 52$.

FIGURE 3 (*B*)

Chromosomes from figure 3 (*A*) arranged as homologous pairs.

FIGURES 4–6

Polar views of equatorial plates at the first meiotic division. Figure 4, *Vulpes;* figure 5, *Alopex-Vulpes* hybrid; figure 6, *Alopex*. All figures × 1800.

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CONTINUED FRACTIONS, ALGEBRAIC FUNCTIONS AND THE PADÉ TABLE

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1. Introduction.—Given a function $f(x)$ possessing a power series development in the neighborhood of the origin, the problem, first posed by Padé, which we wish to consider is that of determining the polynomials $P_{m,n}$; $Q_{m,n}$ the first of degree m, the second of degree n, having the property that

$$P_{m,n}(x) + f(x)Q_{m,n}(x) = x^{m+n+1} + \ldots. \tag{1}$$

The sets $(P_{n,n};\ Q_{n,n})$, $(P_{n,n-1};\ Q_{n,n-1})$ are obtained from the continued fraction expansion

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \sim b_0 + \frac{b_1 x}{1+}\,\frac{b_2 x}{1+}\,\cdots, \tag{2}$$

associated with $f(x)$, cf. Perron.[1]

In general, the problem posed above is difficult despite the fact that the coefficients of the unknown polynomials $P_{m,n}$; $Q_{m,n}$ may be obtained formally by solving the $m + n + 1$ linear equations equivalent to (1). In this way necessary and sufficient conditions may be obtained for the existence of a unique solution to (1). However, this mode of procedure is unsatisfactory in many respects, since it is virtually impossible to apply the criteria found in this way, and the structure of the coefficients b_n, and the polynomials $P_{m,n}$; $Q_{m,n}$ remains hidden.

For these reasons, various artifices have been employed to find the continued fraction expansion of $f(x)$ and the elements $P_{m,n}$; $Q_{m,n}$ of the Padé table. These depend, naturally, upon the particular functional properties of the function under consideration. Thus there are techniques available for functions associated with the hyper-geometric function, for functions satisfying linear differential equations with polynomial coefficients, for functions of the form $\int_{-\infty}^{\infty} d\alpha(t)/(x-t)$ and so on, cf. Perron.[2]

The primary purpose of this paper is to present a method applicable, in principle, to the class of all algebraic functions; in practice, difficulties of various types occur. Nevertheless, using this method we are able to obtain many new continued fraction expansions quite easily. Some examples will be given below.

We also present a method applicable only to algebraic functions defined by equations of the form

$$y^n + p_1(x)y^{n-1} + \ldots + p_n(x) = 0 \tag{3}$$

where the $p_1(x)$ are linear polynomials in x. The most interesting functions of this class are those satisfying the simple equations, $y^n - (1 + x) = 0$.

Let us now sketch the first method. Turning to (1), we note that if $f(x)$ is algebraic, so also is the left-hand side of (1). According to the right-hand side of the equation, this algebraic function must have a zero of order $m + n + 1$ at $x = 0$. Furthermore, the set of poles of this algebraic function is easily determined, consisting of the poles of $f(x)$, which may be found by reference to the defining equation, and the point $x = \infty$. Since an algebraic function is completely determined, apart from a multiplying constant, once its zeroes and poles are known together with their multiplicities, the expression $P_{m,n}(x) + Q_{m,n}(x)f(x)$ will be theoretically known provided that we can extract enough information from (1), the general result for all algebraic functions that the number of zeroes is equal to the number of poles, and the further relations, depending upon the genus of the defining curve, that must hold between the zeroes and the poles.

In some cases, a count of the zeroes and poles is sufficient to fix the left-hand side of (1). In order to furnish an explicit expression, we introduce the transcendental functions which uniformize the curve, e.g., for curves of genus zero, the trigonometric functions, for curves of genus one, the elliptic functions, and so on. One of the practical difficulties is that the transcendental functions, required in general for the uniformization of curves of genus greater than or equal to two, are not readily accessible.

If the initial argument on the zeroes and poles does not suffice, we must utilize the specific form of the left-hand side of (1). An example is given below.

Proofs of the results stated here, and further results, will be published separately.

2. Examples of the First Method.—Consider the curve defined by $y^2 = ax^3 + bx^2 + cx + d$, $a \neq 0$. Let us impose the condition that $ax^3 + bx^2 + cx + d = 0$ have no double root. In this case, the equation of the curve may be written in the form

$$y^2 = 4(c_1x + c_2)^3 - g_2(c_1x + c_2) - g_3, \qquad g_2^3 \neq 27g_3^2, \qquad c_1 \neq 0, \quad (1)$$

and has the parametrization,

$$c_1x + c_2 = p(u), \qquad y = P'(u) \quad (2)$$

where $P(u)$ is the Weierstrassian elliptic function. Let $2w_1$, $2w_2$ be a set of primitive periods of $P(u)$, and α, $-\alpha$ be the roots of $p(u) - c_2 = 0$. Then

THEOREM. *The necessary and sufficient condition for the existence of a unique sequence of polynomials* $\{P_n, Q_{n-1}\}$, P_n *of degree* n, Q_{n-1} *of degree* $n - 1$, *satisfying*

$$P_n(x) + \sqrt{ax^3 + bx^2 + cx + d}\, Q_{n-1}(x) = x^{2n} + \ldots \quad (3)$$

is that

$$\alpha \neq \frac{m_1w_1 + m_2w_2}{n}, \qquad m_1, m_2 = 0, \pm 1, \pm 2, \ldots; \qquad n = 1, 2, \ldots. \quad (4)$$

If this condition is satisfied, we have

$$P_n\left(\frac{p(u) - c_2}{c_1}\right) + p'(u)Q_{n-1}\left(\frac{p(u) - c_2}{c_1}\right) = \frac{c_1^{-2n}p'(\alpha)^{2n}\sigma(\alpha)^{2n+1}\sigma(u + 2n\alpha)\sigma(u - \alpha)^{2n}}{\sigma((2n + 1)\alpha)\sigma(u)^{2n+1}}. \quad (5)$$

From this relation, the elements of the continued fraction expansion for $\sqrt{ax^3 + bx^2 + cx + d}$ *may be derived.*

Sketch of Proof: If we introduce the parametrization (2) then equation (3) becomes

$$P_n\left(\frac{p(u) - c_2}{c_1}\right) + p'(u)\curvearrowright_{n-1}\left(\frac{p(u) - c_2}{c_1}\right) = \left(\frac{p(u) - c_2}{c_1}\right)^{2n} + \ldots. \quad (6)$$

The left-hand side of the above equation is an elliptic function with a pole of order $2n + 1$ at $u = 0$, and, by virtue of the right-hand side, with a zero of order $2n$ at $u = \alpha$. Although $p(u) - c_2 = 0$ has two zeroes, at $u = \pm\alpha$, only one is a zero of the left-hand side, the other being a zero of the conjugate. Since the sum of the zero points must be congruent to the

sum of the points at which the function has poles, multiple points counted multiply, we are led to the expression

$$P_n\left(\frac{p(u)-c_2}{c_1}\right)+p'(u)Q_{n-1}\left(\frac{p(u)-c_2}{c_1}\right)=d_n\sigma(u-\alpha)^{2n}\sigma(u+2n\alpha)/\sigma(u)^{2n+1}. \quad (7)$$

The constant is now easily obtained from the normalization condition that the coefficient of x^{2n} in (3) be unity.

Once the sequence $\{P_n, Q_{n-1}\}$ has been obtained, an elementary result from the theory of continued fractions permits us to deduce the analogous sequence where both polynomials have the same degree, and thence the elements of the continued fraction.

As a simple illustration of the case where a count of the zeroes and poles is not sufficient, let us consider the function $y=(1+x)\sqrt{1-x^2}$. Using the obvious parametric representation, the problem takes the form of determining the trigonometric polynomials P_n, Q_{n-1} such that

$$P_n(\sin\theta)+Q_{n-1}(\sin\theta)(1+\sin\theta)\cos\theta=(\sin\theta)^{2n}+\ldots. \quad (8)$$

Here the left side of the equation is a trigonometric polynomial of degree $n+1$, possessing, in consequence, $2n+2$ zeroes, of which only $2n$ are determined by the equation. Therefore,

$$P_n(\sin\theta)+Q_{n-1}(\sin\theta)(1+\sin\theta)\cos\theta=(1-\cos\theta)^nA_1(\theta) \quad (9)$$

where $A_1(\theta)$ is a trigonometric polynomial of the first degree. From this we obtain, via the substitution $\theta\rightarrow\pi-\theta$

$$P_n(\sin\theta)=[(1-\cos\theta)^nA_1(\theta)+(1+\cos\theta)^nA_1(\pi-\theta)]/2,$$
$$(1+\sin\theta)Q_{n-1}(\sin\theta)=[(1-\cos\theta)^nA_1(\theta)-(1+\cos\theta)^nA_1(\pi-\theta)]/2\cos\theta. \quad (10)$$

The unknown linear polynomial is now readily found.

We remark, however, that the sequence $\{P_{n,\,n-2},\ Q_{n,\,n-2}\}$ can be determined by the first method. In general, for every algebraic function there is an integer k such that the sequence of polynomials $\{P_{n,\,n-k},\ Q_{n,\,n-k}\}$ is determined by the first method.

A similar, but more complicated investigation is required to derive the continued fraction associated with $y^2=ax^4+bx^2+cx^3+dx+e$. This method is applicable to the treatment of N-dimensional continued fractions, but lack of space does not permit any example of this here.

3. An Example of the Second Method.—We seek to determine the polynomials $P_{n,\,i}$, $i=1, 2, 3$, each of degree n, satisfying the equation

$$P_{n,1}+P_{n,2}(1+x)^{1/3}+P_{n,3}(1+x)^{2/3}=x^{3n+2}+\ldots. \quad (1)$$

It can be shown that these polynomials are unique. Choose A_1, a linear polynomial, and A_2, A_3 constants, so that

$$A_1 + A_2(1+x)^{1/3} + A_3(1+x)^{2/3} = x^3 + \ldots. \tag{2}$$

Multiplying (1) and (2), and taking account of the uniqueness of the $P_{n+1,i}$ we obtain the set of linear difference equations,

$$\begin{aligned} P_{n+1,1} &= A_1P_{n,1} + A_3(1+x)P_{n,2} + A_2(1+x)P_{n,3} \\ P_{n+1,2} &= A_2P_{n,1} + A_1P_{n,2} + A_3(1+x)P_{n,3} \\ P_{n+1,3} &= A_3P_{n,1} + A_2P_{n,2} + A_1P_{n,3}. \end{aligned} \tag{3}$$

Since the initial polynomials, $P_{0,i}$, are easily found, the $P_{n,i}$ are determined once the characteristic roots of the coefficient matrix

$$A = \begin{pmatrix} A_1 & A_3(1+x) & A_2(1+x) \\ A_2 & A_1 & A_3(1+x) \\ A_3 & A_2 & A_1 \end{pmatrix} \tag{4}$$

are known. This matrix resembles a circulant, and it is easy to verify that the characteristic roots are given by

$$\begin{aligned} \lambda_1 &= A_1 + A_2\sqrt[3]{1+x} + A_3\sqrt[3]{(1+x)^2} \\ \lambda_2 &= A_1 + \omega A_2\sqrt[3]{1+x} + \omega^2A_3\sqrt[3]{(1+x)^2} \\ \lambda_3 &= A_1 + \omega^2A_2\sqrt[3]{1+x} + \omega A_3\sqrt[3]{(1+x)^2}, \end{aligned} \tag{5}$$

where ω is a primitive cubic root of unity.

These results generalize immediately to the N-dimensional case. The above formulas exhibit the known correspondence between the three-dimensional continued fraction and the decomposable cubic form

$$x^3 + y^3 + z^3 + 3xyz = (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z) \tag{6}$$

cf. Daus.[1]

The method sketched here for the three-dimensional case applies to those algebraic functions satisfying (3) of section 1.

[1] Daus, P. H., "Normal Ternary Continued Fraction Expansions for the Cube Roots of Integers," *Am. J. Math.*, **44**, 279 (1922).

[2] Perron, O., *Die Lehre von den Kettenbrüchen*, Chap. X, Leipzig, 1929.

[3] Whittaker, E. T., and Watson, G. N., *A Course of Modern Analysis*, chapters VIII, IX and X.

ON THE NUMBER OF SOLUTIONS OF SOME TRINOMIAL EQUATIONS IN A FINITE FIELD

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In the present paper we shall use the results in two previous papers[1] by the authors to obtain (Theorem I) the exact number, explicitly in terms of p, n, k_1 and k_2, of solutions of the equation, if $0 < a_i < p^n - 1$, $(p^n - 1, a_i) = k_i$, $i = 1, 2$;

$$c_1x_1^{a_1} + c_2x_2^{a_2} + c_3 = 0, \tag{1}$$

in x_1 and x_2, non-zero elements of a finite field $F(p^n)$ of order p^n, p an odd prime with c_1, c_2, c_3 given elements of $F(p^n)$, $c_1c_2c_3 \neq 0$, in certain special cases. Secondly, we shall find limits (Theorem II) for the number of solutions of (1) which are better than those given in C for the solutions of this particular equation, and which have the unusual property that they agree with the exact values found for the number of solutions in the special cases treated in Theorem I. We shall employ the notation used in H (defined in (9b) just below relation (14) of that paper).

Let α be a primitive (ds_1s_2)-th root of unity, $m = ds_1s_2$ where $ds_1 = k_1$, $ds_2 = k_2$ where $(k_1, k_2) = d$, whence $(s_1, s_2) = 1$. Let χ_{k_1} be a special k_1-character and χ_{k_2} a special k_2-character in $F(p^n)$ and write, if K stands for $F(p^n)$,

$$\tau(\chi_{k_1}^{\mu_1}) = \sum_{a \in K} \alpha^{\mu_1 s_2 \operatorname{ind}(a)} \zeta^{tr(a)}$$

and

$$\tau(\chi_{k_2}^{\mu_2}) = \sum_{a \in K} \alpha^{\mu_2 s_1 \operatorname{ind}(a)} \zeta^{tr(a)},$$

where $tr(a)$ is the trace of a in $F(p^n)$ with respect to $F(p)$, and $g^{\operatorname{ind}(a)}$ is defined by

$$g^{\operatorname{ind}(a)} = a$$

in $F(p^n)$, g being a primitive root of $F(p^n)$ and we define ind (0) as zero. Hence, as is known, provided $\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2} \neq 1$,

$$\frac{\tau(\chi_{k_1}^{\mu_1})\tau(\chi_{k_2}^{\mu_2})}{\tau(\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2})} = \sum_a \alpha^{\mu_1 s_2 \operatorname{ind}(a) + \mu_2 s_1 \operatorname{ind}(1-a)}.$$

Also, setting $(-a_1)$ for (a) we obtain

$$\sum_{a \in K} \alpha^{\mu_1 s_2 \text{ ind } (a) + \mu_2 s_1 \text{ ind } (1-a)}$$

$$= \sum_{a_1 \in K} \alpha^{\mu_1 s_2 \text{ ind } (-a_1) + \mu_2 s_1 \text{ ind } (1+a_1)}$$

$$= \sum_{a_1 \in K} \alpha^{\mu_1 s_2 \text{ ind } (-1)} \cdot \alpha^{\mu_1 s_2 \text{ ind } (a_1) + \mu_2 s_1 \text{ ind } (1+a_1)}.$$

Let $n = 2n_1$, then $p^{2n_1} - 1 \equiv 0 \pmod 4$ and if $(\mu_1 s_2(p^{2n_1} - 1))/2 = e$ so that $\alpha^{\mu_1 s_2 \text{ ind } (-1)} = (-1)^e = 1$,

$$\frac{\tau(\chi_{k_1}^{\mu_1})\tau(\chi_{k_2}^{\mu_2})}{\tau(\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2})} = \sum_{a \in K} \alpha^{\mu_1 s_2 \text{ ind } (a) + \mu_2 s_1 \text{ ind } (a+1)} = \psi(\alpha) \qquad (2)$$

Now Mitchell[2] showed that if $\mu_1 s_2 \not\equiv 0 \pmod m$, $\mu_2 s_1 \not\equiv 0 \pmod m$ and $\mu s_2 + \mu' s_1 \not\equiv 0 \pmod m$ then

$$\psi(\alpha) = (-1)^{r-1} p^{n_1}, \qquad (3)$$

where p belongs to the exponent $2t$, modulo m and $n_1 = rt$ with $p^t \equiv -1 \pmod m$. We now employ (18) of H, which may be written as follows:

$$N = p^n - 2 - \sum_{\mu_1=0}^{k_1-1} \chi_{k_1}^{\mu_1}\left(-\frac{c_3}{c_1}\right) - \sum_{\mu_2=0}^{k_2-1} \chi_{k_2}^{\mu_2}\left(-\frac{c_3}{c_2}\right) - \sum_{\nu=1}^{d-1} \chi_{k_1}^{\nu_1}\left(-\frac{c_2}{c_1}\right)$$

$$+ \sum_{\mu_1, \mu_2} \chi_{k_1}^{\mu_1}\left(-\frac{c_3}{c_1}\right) \chi_{k_2}^{\mu_2}\left(-\frac{c_3}{c_2}\right) \frac{\tau(\chi_{k_1}^{\mu_1})\tau(\chi_{k_2}^{\mu_2})}{\tau(\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2})}, \qquad (4)$$

where in the last term $\chi_{k_1}^{\mu_1} \neq 1$, $\chi_{k_2}^{\mu_2} \neq 1$, $\chi_1^{\mu_1}\chi_2^{\mu_2} \neq 1$, and divide the discussion into seven cases in order to express s in terms of integers only. Because of (3) and $\chi(\alpha) = -1$ for $\mu_1 = 0$, $\mu_2 \neq 0$, also for $\mu_1 \neq 0$, $\mu_2 = 0$, this is always possible.

The cases are:

I. $c_3 = c_2 = c_1$. II. $c_3 = c_1$, $c_2 \neq c_1$ with ind $c_2 \equiv$ ind $c_1 \pmod d$. III. $c_3 = c_1$, ind $c_2 \not\equiv$ ind $c_1 \pmod d$. IV. $c_3 = c_2$, $c_1 \neq c_2$, ind $c_2 \equiv$ ind $c_1 \pmod d$. V. $c_3 = c_2$, $c_2 \not\equiv c_1 \pmod d$. VI. $c_3 \neq c_2$, $c_3 \neq c_1$, ind $c_2 \equiv$ ind $c_1 \pmod d$. VII. $c_3 \neq c_1$, $c_3 \neq c_2$, $c_1 \not\equiv c_2 \pmod d$. Write

$$A = \sum_{\mu_1, \mu_2} \chi_{k_1}^{\mu_1}\left(-\frac{c_3}{c_1}\right) \chi_{k_2}^{\mu_2}\left(-\frac{c_3}{c_2}\right) \frac{\tau(\chi_{k_1}^{\mu_1})\tau(\chi_{k_2}^{\mu_2})}{\tau(\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2})} \qquad (5)$$

where $\chi_{k_1}^{\mu_1} \neq 1$, $\chi_{k_2}^{\mu_2} \neq 1$, $\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2} \neq 1$.

In case I, we find

$$\sum_{\mu_1=0}^{k_1-1} \chi_{k_1}^{\mu_1}\left(-\frac{c_3}{c_1}\right) = k_1,$$

$$\sum_{\mu_2=0}^{k_2-1} \chi_{k_2}^{\mu_2}\left(-\frac{c_3}{c_2}\right) = k_2,$$

$$\sum_{s=1}^{d-1} \chi_{k_1}^{s\nu}\left(-\frac{c_2}{c_1}\right) = d - 1.$$

To reduce A in this case, we take all the terms in which $\chi_{k_1}^{\mu_1} \neq 1$, $\chi_{k_2}^{\mu_2} \neq 1$ and then subtract all the terms from the result in which $\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2} = 1$ with $\chi_{k_1}^{\mu_1} \neq 1$, $\chi_{k_2}^{\mu_2} \neq 1$ and employ (3). The terms of the first type give

$$(k_1 - 1)(k_2 - 1)p^{n_1}(-1)^{r-1}.$$

As in the proof of (18) of s there are $(d - 1)$ terms of the second type just mentioned. Hence (4) reduces to

$$p^n + 1 - k_1 - k_2 - d + ((k_1 - 1)(k_2 - 1) - (d - 1))(-1)^{r-1}p^{n_1}. \tag{6}$$

For case II, we find the same value for the first summation in (4) but for the second, we have the value 0 since $c_3 \neq c_2$. In reducing A for this case and separating the summation in the same way it was separated in case I, we have for the first part $(k_1 - 1)p^{n_1}(-1)^r$ and for the second part $(d - 1)p^{n_1}(-1)^r$. Hence, we have for case II the value, if N_i denotes the number of solutions in case i, $i = \text{I, II, } \ldots, \text{VII}$.

$$N_2 = p^n + 1 - d - k_1 + (k_1 + d - 2)p^{n_1}(-1)^r. \tag{7}$$

Similarly for cases III, IV, etc., we have

$$N_3 = p^n + 1 - k_1 + (k_1 - 2)p^{n_1}(-1)^r. \tag{8}$$

$$N_4 = p^n + 1 - d - k_2 + (k_2 + d - 2)p^{n_1}(-1)^r. \tag{9}$$

$$N_5 = p^n + 1 - k_2 + (k_2 - 2)p^{n_1}(-1)^r. \tag{10}$$

$$N_6 = p^n + 1 - d + p^{n_1}(-1)^r(d - 2). \tag{11}$$

$$N_7 = p^n + 1 + (-1)^{r-1}2p^{n_1}. \tag{12}$$

THEOREM I. *If N_i is the number of solutions of the equation*

$$c_1x_1^{a_1} + c_2x_2^{a_2} + c_3 = 0$$

in x_1 and x_2, non-zero elements of a finite field $F(p^n)$ of order p^n, p an odd prime, with c_1, c_2, c_3 given non-zero elements of $F(p^n)$ with N_i, $i = 1, 2, \ldots, 7$, the number of solutions in the corresponding seven cases numbered I, II, ..., VII, given just above the relation (5); $0 < a_i < p^n - 1$, $(a_i, p^n - 1) = k_i$, $i = 1, 2$; $m = ds_1s_2$, $ds_1 = k_1$, $ds_2 = k_2$, $(k_1, k_2) = d$; $n = 2n_1$, p belongs to the exponent $2t$ modulo m, $n_1 = rt$ with $p^t \equiv -1 \pmod m$, then the values of N_i are given by the relations (6)–(12), inclusive.

Mitchell[2] obtained these results for the special case when $k_1 = k_2$ by a more complicated method.

We now use (4) to find closer limits for N in the special case when $s = 2$, than those found in C for equation (1) of that paper. Set

$$M = p^n \sum_{\mu_1=0}^{k_1-1} \chi_{k_1}^{\mu_1}\left(-\frac{c_3}{c_1}\right) - \sum_{\mu_2=0}^{k_2-1} \chi_{k_2}^{\mu_2}\left(-\frac{c_3}{c_2}\right) - \sum_{\tau=1}^{d-1} \chi_{k_1}^{\tau s_1}\left(-\frac{c_2}{c_1}\right).$$

Now every term on the right of (4) is real so that we may write $N \leqq M + |A|$ and $N \geqq M - |A|$ and using the known result

$$|\tau(\chi)| = p^{n/2}, \ \chi \neq \chi_0, \tag{13}$$

we have, if

$$D = |((k_1 - 1)(k_2 - 1) - (d - 1))p^{n/2}|, \tag{13a}$$

$$N \leqq M + D, \tag{14}$$

$$N \geqq M - D. \tag{15}$$

To obtain limits in terms of rational integers only, divide the discussion into the seven cases I–VII defined in the proof of Theorem I. Then we can evaluate M exactly as in the determination of N in Theorem I in each of the seven cases.[4]

Write N_i; $i = 1, 2, \ldots, 7$ for the value of N corresponding to each of the seven cases referred to in Theorem I, respectively. Then from (14) and (15) we have the

THEOREM II. *If*

$$c_1x_1^{a_1} + c_2x_2^{a_2} + c_3 = 0,$$

where c_1, c_2, c_3 are given elements in a finite field $F(p^n)$ of order p^n, p prime, $c_1c_2c_3 \neq 0$, then we have the following limits for N_i, the number of solutions in the x's, neither zero, of the above equation,

$$M_i - D \leqq N_i \leqq M_i + D, \tag{16}$$

where $M_1 = p^n + 1 - k_1 - k_2 - d$; $M_2 = p^n + 1 - d - k_1$; $M_3 = p^n + 1 - k_1$; $M_4 = p^n + 1 - d - k_2$; $M_5 = p^n + 1 - k_2$; $M_6 = p^n + 1 - d$; $M_7 = p^n + 1$; $k_1 = (p^n - 1, a_1)$, $k_2 = (p^n - 1, a_2)$, $d = (k_1, k_2)$ and D is defined in (13a), M_i; $i = 1, 2, \ldots, 7$, is the value of M corresponding to each of the seven cases, I, II, ..., VII, of Theorem I, respectively.

For the case when $n = 1$ and $k_1 = k_2$ is an odd prime, Hurwitz[3] by a different method proved the above theorem using rational integers only. His argument is much longer than ours. For the particular cases treated in Theorem I the *exact value for N agrees with the limits found in Theorem II, when $c_1 = c_2 = c_3$, with the superior limit when r is odd and the inferior limit when r is even.* For the case $n = 1$, $k_1 = k_2$ an odd prime, this fact was noted by Mitchell,[2] who also stated that Hurwitz's method could be extended so as to obtain limits for the case when n is general, which had a similar property.

[1] These PROCEEDINGS, **35,** 94–99 (1949), this will be referred to as C; **35,** 451–457 (1949), this will be referred to as H.

[2] *Ann. Math.*, **II,** 18, 120 (1917).

[3] *Crelle*, **136,** 272–292 (1909).

[4] As a check on the accuracy of our work, the present problem was discussed in a mathematical seminar, using a different method of proof. During the resulting discussion, Mr. Olin B. Faircloth made a suggestion, which, when used in connection with the method employed in this paper yielded a better superior limit for N, than that which we originally had.

ON THE NATURE OF THE SOLUTIONS OF CERTAIN EQUATIONS IN A FINITE FIELD

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In two recent papers[1] the authors considered the equation

$$c_1x_1^{a_1} + c_2x_2^{a_2} + \ldots + c_sx_s^{a_s} + c_{s+1} = 0 \qquad (1)$$

in the x's, where the a's are integers such that $0 < a < p^n - 1$; $s \geqq 2$ for $c_{s+1} \not\equiv 0$, and $s > 2$ for $c_{s+1} = 0$ the c's being given elements of a finite field of order p^n, p an odd prime, which will be designated by $F(p^n)$; and

$$c_1c_2 \ldots c_sx_1x_2 \ldots x_s \not\equiv 0 \qquad (2)$$

in $F(p^n)$. Limits were found for the number of solutions of (1) and it was proved that for p^n sufficiently large (1) always had solutions under the restriction (2). In the statement above concerning (1) we made the provision that $s \geqq 2$ for $c_{s+1} \not\equiv 0$. If we consider the case where $s = 1$, (1) reduces to the equation

$$c_1x_1^{a_1} + c_2 = 0, \qquad (3)$$

with $c_1c_2 \not\equiv 0$. Since any finite field can be represented by means of the residue classes with respect to some prime ideal in an algebraic field, then examination of the solutions of (3) is included in the theory of the congruence

$$x^n \equiv \alpha \pmod{\mathfrak{p}}, \qquad (4)$$

where α is an integer in an algebraic field K, $\mathfrak{p}$ is a prime ideal in that field and $\alpha \not\equiv 0 \pmod{\mathfrak{p}}$. The study of the last relation led to the theory of the class field and the laws of reciprocity for nth powers.[2]

We wish to point out the sharp distinction between this problem and the problem of finding the solutions or the number of solutions of the equation (1) and when $s \geqq 2$ for $c_{s+1} \not\equiv 0$.

In addition to this, we shall also obtain an expression (Theorem I) for the number of solutions of (1) when $c_{s+1} \not\equiv 0$ and $s = 2$, which leads to the proof of Theorem I and which we shall also have occasion to make use of in another paper. Also we obtain Theorem II, which gives the exact number of solutions of (1) when the k's are prime each to each, $c_{s+1} = 0$ and $(a_i, p^n - 1) = k_i$; $i = 1, 2, \ldots, s$.

We consider equation (4) and confine ourselves to the case where K is an algebraic field which contains a primitive lth root of unity with l prime. Then

$$x^l - \alpha \equiv 0 \pmod{\mathfrak{p}} \tag{5}$$

if and only if the lth power character of α, modulo $\mathfrak{p}$ is unity, or

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_l = 1.$$

However, by a known theorem[3]

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_l \neq 1$$

for an infinity of distinct prime ideals $\mathfrak{p}$ provided α does not equal an lth power of an integer in K. Now the last statement is contrary to what we have for $s > 1$ in (1) with $c_{s+1} \not\equiv 0$. For, as we have shown in E (p. 262) the congruence

$$\alpha_1 x_1^{a_1} + \alpha_2 x_2^{a_2} + \ldots + \alpha_s x_s^{a_s} + \alpha_{s+1} \equiv 0 \pmod{\mathfrak{p}}, \tag{6}$$

with

$$\alpha_1 \alpha_2 \ldots \alpha_s x_1 x_2 \ldots x_s \not\equiv 0 \pmod{\mathfrak{p}}$$

for $s \geqq 2$ with $c_{s+1} \not\equiv 0 \pmod{\mathfrak{p}}$ always has solutions if the norm of $\mathfrak{p}$ exceeds a certain limit, in fact (6) always has at least k solutions in the x's for k any positive integer if the norm of p exceeds a certain limit. In particular when we consider the case where there are no solutions it is known[4] that $x^7 + y^7 + 1 \equiv 0 \pmod{p}$ has no solutions in integers prime to p for just four primes p, namely 29, 71, 113, 491, but has solutions for all the other primes p. As for $x^5 + y^5 + 1 \equiv 0 \pmod{p}$ it has no solutions for $p = 11, 41, 71, 101$. Also $x^3 + y^3 + 1 \equiv 0 \pmod{p}$ has no solutions only for $p = 7$ and 13. It is also known[5] that for $m < 10l$, and $p = 1 + ml$ and $h(l)$ is the number of primes p for which

$$x^l + y^l + 1 \equiv 0 \pmod{p} \tag{7}$$

has no solutions for $xy \not\equiv 0 \pmod{p}$ then $h(11) = 3$, $h(13) = 4$, $h(17) = 4$,

$h(19) = 6$. In view of these facts it may happen that $h(l)$ increases as l increases. Note also from the cases 3, 5 and 7 that there is a range of values for p in which the corresponding congruences may or may not have solutions.

From the above considerations we may say that associated with any form of the type given by the left-hand member of (4) there is an integer $e(k)$ such that there are exactly $e(k)$ values of $\mathfrak{p}$ such that (4) has k solutions. Even in the case where k is zero, the class of forms satisfying the condition may be rather general. For example,[6] it was shown that if c is a prime and m is an integer such that $p = 1 + mc$ with p prime then

$$a_1x_1^m + a_2x_2^m + \ldots + a_sx_s^m \equiv 0 \pmod{p}$$

has only the solutions $x_1 \equiv x_2 \equiv \ldots \equiv x_s \equiv 0 \pmod{p}$ provided $s \leqq c - 2$. the sum of no n of the a's is $\equiv 0 \pmod{p}$ $0 < n \leqq s$, and

$$(|a_1| + |a_2| + \ldots + |a_s|)^{\phi(c)} < p,$$

where $\phi(c)$ is the indicator of c. Select $a_1 = a_2 = \ldots = a_{s-1} = 1$ and $a_s = -s$, and also an m and p with $p = 1 + mc$ with c fixed and satisfying the other conditions just mentioned then it is evident that

$$x_1^m + x_2^m + \ldots + x_{s-1}^m - sx_s^m \equiv 0 \pmod{p}$$

has no primitive solutions. (Incidentally, in view of Theorem III of the article just referred to, the equation

$$x_1^m + x_2^m + \ldots + x_{s-1}^m - sx_s^m = 0$$

has no solutions in rational integers unless each x is zero, if there is a prime p with $p = 1 + mc$ and $(2s - 1)^{\phi(c)} < p$.)

We now recur to the subject of the number of solutions of equation (1), treated in E and C. The case $s = 2$ with $c_3 \neq 0$, has received considerable atten'' n in the literature.[7] Let us examine

$$c_1x_1^{a_1} + c_2x_2^{a_2} + c_3 = 0, \tag{8}$$

with

$$x_1x_2c_1c_2c_3 \neq 0. \tag{8a}$$

Theorem I of C, we see will give, if N is the number of solutions of (8) in x_1 and x_2 under the conditions (8a),

$$N = \frac{(p^n - 1)^2}{p^n} + \frac{T}{p^n}, \tag{9}$$

where we may write T in the form, if $k_i = (a_i, p - 1)$, μ_1 ranges over 0, 1, ..., $k_1 - 1$; μ_2 over 0, 1, ..., $k_2 - 1$

$$\sum_{\substack{a \in \mathbf{K} \\ a \neq 0}} \sum_{\mu_1, \mu_2} \tau(\chi_{k_1}^{\mu_1})\tau(\chi_{k_2}^{\mu_2})\chi_{k_1}^{\mu_1}(c_1^{-1})\chi_{k_2}^{\mu_2}(c_2^{-1})\chi_{k_1}^{\mu_1}(a^{-1})\chi_{k_2}^{\mu_2}(a^{-1})\zeta^{Tr(ac_3)}, \tag{9a}$$

where χ_{k_1} is a special character defined by k_1 or as we shall say, a k_1-character, and similarly χ_{k_2} is a special character defined by k_2 and where, if we write K for $F(p^n)$,

$$\tau(\chi, \zeta) = \tau(\chi) = \sum_{b \in K} \chi(b)\zeta^{tr(b)}; \quad \zeta = e^{2i\pi/p}, \tag{9b}$$

with $\chi(0) = 0$. This is obtained from the fact that if α is a primitive $(p^n - 1)$th root of unity then $\alpha^{\mu \text{ ind } a}$, where a ranges over all elements of $F(p^n)$, $\neq 0$, gives a character as defined in C where ind a is given by

$$g^{\text{ind } a} = a,$$

and where g is a primitive root in $F(p^n)$, so we obtain all characters by taking $\mu = 0, 1, \ldots, p^n - 2$. We have from C the relations, if χ_0 is the principal character,

$$\left.\begin{aligned} |\tau(\chi)| &= p^{n/2}; \ \chi \neq \chi_0 \\ \tau(\chi_0) &= -1 \end{aligned}\right\}. \tag{10}$$

To reduce (9a) we divide the discussion into four cases.

First Case. $\chi_{k_1}{}^{\mu_1} = 1, \chi_{k_2}{}^{\mu_2} = 1$. In this event $\chi_{k_1}{}^{\mu_1}(a^{-1}) = \chi_{k_2}{}^{\mu_2}(a^{-1}) = 1, \chi_{k_1}{}^{\mu_1}(c_1{}^{-1}) = \chi_{k_2}{}^{\mu_2}(c_2{}^{-1}) = 1$, which reduces (9a), using (10), to

$$\sum_a \zeta^{tr(ac_3)} = -1. \tag{11}$$

Second Case. $\chi_{k_1}{}^{\mu_1} = 1; \ \chi_{k_2}{}^{\mu_2} \neq 1$. This gives from (9a) and (10),

$$-\sum_a \tau(\chi_{k_2}{}^{\mu_2})\chi_{k_2}{}^{\mu_2}(c_2{}^{-1}a^{-1})\zeta^{tr(ac_3)} = \sum_a -\tau(\chi_{k_2})\chi_{k_2}(c_2{}^{-1})(\chi_{k_2}(a))^{-1}\zeta^{tr(ac_3)},$$

and setting $ac_3 = -a_1$, using $c_3 \neq 0$, this reduces to

$$-\sum_{a_1} \tau(\chi_{k_2}{}^{\mu_2})\chi_{k_2}{}^{\mu_2}(-c_2{}^{-1}c_3)(\chi_{k_2}{}^{\mu_2}(a_1))^{-1}\zeta^{-tr\,a_1} = -\sum \tau(\chi_{k_2})\overline{\tau(\chi_{k_2})}\chi_{k_2}\left(-\frac{c_3}{c_2}\right),$$

and using the known relation

$$\tau(\chi_{k_2}{}^{\mu_2})\overline{\tau(\chi_{k_2}{}^{\mu_2})} = p^n, \tag{11a}$$

this is seen to equal

$$-p^n \sum_{\mu_2} \chi_{k_2}{}^{\mu_2}\left(-\frac{c_3}{c_2}\right). \tag{12}$$

Also, if $\chi_{k_1}{}^{\mu_1} \neq 1$ and $\chi_{k_2}{}^{\mu_2} = 1$ we find in a similar way that the result is

$$-p^n \sum_{\mu_1} \chi_{k_1}{}^{\mu_1}\left(-\frac{c_3}{c_1}\right). \tag{13}$$

Third Case. $\chi_{k_1}{}^{\mu_1}\chi_{k_2}{}^{\mu_2} = 1$ with $\chi_{k_1}{}^{\mu_1} \neq 1$, $\chi_{k_2}{}^{\mu_2} \neq 1$, (9a) reduces to

$$\sum_{a}\sum_{\mu_1}\chi_{k_1}^{\mu_1}(c_1^{-1})\chi_{k_1}^{-\mu_1}(c_2^{-1})\tau(\chi_{k_1}^{\mu_1})\tau(\chi_{k_1}^{-\mu_1},\zeta)\zeta^{-tr(ac_3)}.$$

Now

$$\begin{aligned}\tau(\chi_{k_1}^{-\mu_1},\zeta) &= \overline{\tau(\chi_{k_1}^{-\mu_1},\zeta^{-1})}\chi_{k_1}(-1)\\ &= \overline{\tau(\chi_{k_1}^{\mu_1})}\chi_{k_1}^{\mu_1}(-1).\end{aligned}$$

Hence for the admissible values of μ_1, (9a) reduces to

$$-\sum_{\mu_1}\chi_{k_1}^{\mu_1}\left(-\frac{c_2}{c_1}\right)p^n. \tag{14}$$

Now we must determine the possible values of μ_1 from $\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2}=1$. Let α be a primitive (ds_1s_2)th root of unity where $ds_1=k_1$, $ds_2=k_2$ and where $(k_1,k_2)=d$, whence $(s_1,s_2)=1$. Then $\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2}=1$ gives integers μ_1 and μ_2 such that $\alpha^{s_1\mu_1+s_2\mu_2}=1$ or $s_2\mu_1+s_1\mu_2\equiv 0 \pmod{(ds_1s_2)}$ so that $\mu_1=v_1s_1$, $\mu_2=v_2s_2$. Hence the possible values of μ_1 are s_1v_1; $v_1=1,2,\ldots,d-1$, as the case $v_1=0$ was excepted. Hence (9a) reduces to, for this case,

$$-\sum_{v_1=1}^{d-1}\chi_{k_1}^{s_1v_1}\left(-\frac{c_2}{c_1}\right)p^n. \tag{15}$$

Fourth Case. Here $\chi_{k_1}^{\mu_1}\neq 1$, $\chi_{k_2}^{\mu_2}\neq 1$, $\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2}\neq 1$. Consider such terms in (9a). They may be written in the form

$$\sum_{\mu_1,\mu_2}\sum_{a\in K}\tau(\chi_{k_1}^{\mu_1})\tau(\chi_{k_2}^{\mu_2})\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2}(a^{-1})\zeta^{-tr(ac_3)}\cdot\chi_{k_1}^{\mu_1}(c_1^{-1})\chi_{k_2}^{\mu_2}(c_2^{-1}). \tag{16}$$

Write $a_1=ac_3$, then $a^{-1}=a_1^{-1}c_3$ and using $c_3\neq 0$,

$$\begin{aligned}\sum_{\substack{a_1\in K\\ a_1\neq 0}}\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2}(a^{-1}) &= \sum_{a_1\in K}\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2}(c_3)(\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2}(a))^{-1}\zeta^{-tr(a_1)}\\ &= \sum_{a_1\in K}\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2}(c_3)(\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2}(a_1))^{-1}\chi_{k_1}^{\mu_2}\chi_{k_2}^{\mu_2}(-1)\zeta^{-tr(a_1)}\end{aligned}$$

Hence using (11a), we find from (9a) that the terms in case 4 in T may be written as

$$p^n\sum_{\mu_1,\mu_2}\chi_{k_1}^{\mu_1}\left(-\frac{c_3}{c_1}\right)\chi_{k_2}^{\mu_2}\left(-\frac{c_3}{c_2}\right)\frac{\tau(\chi_{k_1}^{\mu_1})(\chi_{k_2}^{\mu_2})}{\tau(\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2})}, \tag{17}$$

$\chi_{k_1}^{\mu_1}\neq 1$, $\chi_{k_2}^{\mu_2}\neq 1$, $\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2}\neq 1$. Using (9), (9a), (11), (12), (13), (14), (15) and (17), we find

$$\begin{aligned}N = p^n &- \sum_{\mu_1=0}^{k_1-1}\chi_{k_1}^{\mu_1}\left(-\frac{c_3}{c_1}\right)-\sum_{\mu_2=0}^{k_2-1}\chi_{k_2}^{\mu_2}\left(-\frac{c_3}{c_2}\right)-\sum_{v=1}^{d-1}\chi_{k_1}^{s_1v}\left(-\frac{c_2}{c_1}\right)\\ &+\sum_{\mu_1,\mu_2}'\chi_{k_1}^{\mu_1}\left(-\frac{c_3}{c_1}\right)\chi_{k_2}^{\mu_2}\left(-\frac{c_3}{c_2}\right)\frac{\tau(\chi_{k_1}^{\mu_1})\tau(\chi_{k_2}^{\mu_2})}{\tau(\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2})}\end{aligned} \tag{18}$$

where, in the last term, $\chi_{k_1}^{\mu_1}\neq 1$, $\chi_{k_2}^{\mu_2}\neq 1$, $\chi_{k_1}^{\mu_1}\chi_{k_2}^{\mu_2}\neq 1$.

Whence we have the (cf. Davenport-Hasse[8])

THEOREM I. *If N is the number of solutions in x_1 and x_2 of*

$$c_1x_1^{a_1} + c_2x_2^{a_2} + c_3 = 0,$$

with c_1, c_2, c_3, x_1, x_2 in a finite field $F(p^n)$ of order p^n with p an odd prime, $c_1c_2c_3x_1x_2 \neq 0$ in $F(p^n)$; $(a_1, p^n - 1) = k_1$, $0 < k_1 < p^n - 1$, $(a_2, p^n - 1) = k_2$, $0 < k_2 < p^n - 1$, χ_{k_1} is a special k_1-character, χ_{k_2} is a special k_2-character, $d = (k_1, k_2)$, $ds_1 = k_1$, $\tau(\chi)$ is defined as in (9b), then the relation (18) holds.

We shall now obtain N explicitly in terms of p, n and s, for special values of k_1 when $c_{s+1} = 0$. By the remark immediately following (9b), we may write

$$\chi_{k_i}^{\mu_i}(a^{-1}) = \alpha_i^{\mu_i \operatorname{ind}(a^{-1})} \tag{19}$$

for any i in the set $0, 1, \ldots, k_i - 1$, and where α_i is a primitive k_ith root of unity. Set in Theorem I of C, $c_{s+1} = 0$ then for this case we have

$$T = \sum_{\mu_i=0}^{k_i-1} \sum_{\substack{a \in K \\ a \neq 0}} (\alpha_1^{\mu_1} \alpha_2^{\mu_2} \ldots \alpha_s^{\mu_s})^{\operatorname{ind}(a^{-1})} \cdot \prod_{i=1}^{s} \alpha_i^{\mu_i \operatorname{ind}(c_i^{-1})} \tau(\alpha_i^{\mu_i}). \tag{20}$$

Assume that the k's are prime each to each and carry out the summation with respect to a we obtain

THEOREM II. *If N_s is the number of solutions of (1) in x's under the conditions (2) and the other conditions stated in connection with (1), and also $k_1, k_2, \ldots, k_s$ are prime each to each with $c_{s+1} = 0$, then*

$$N_s = \frac{p^n - 1}{p^n} ((p^n - 1)^{s-1} + (-1)^s). \tag{21}$$

This result may be proved in a simpler way as we shall now show.

Consider the equation

$$c_1x_1^{k_1} + \ldots + c_sx_s^{k_s} = 0, \quad (k_i, k_j) = 1, \quad i \neq j, \tag{22}$$

and $A = k_1k_2 \ldots k_i$ dividing $p^n - 1$. Since $(k_1, k_2, \ldots k_s) = 1$, we have integers λ and μ such that

$$\lambda k_1 + \mu k_2 \ldots k_s = 1, \qquad (\lambda, p^n - 1) = 1.$$

Putting

$$x_1 = y_1^{\lambda}, \quad x_2 = y_1^{-\mu A/k_1k_2}y_2, \quad x_3 = y_1^{-\mu A/k_1k_2}y_3, \ldots$$

Then we have

$$c_1y_1^{\lambda k_1} + y_1^{-\mu A/k_1}(c_2y_2^{k_2} + \ldots + c_sy_s^{k_s}) = 0$$

i.e.,

$$c_1y_1 + c_2y_2^{k_2} + \ldots + c_sy_s^{k_s} = 0$$

Since, for given elements of $F(p^n)$, $y_2, \ldots, y_s$ such that

$$c_2 y_2^{k_2} + \ldots + c_s y_s^{k_s} \neq 0$$

we have a unique y_1, therefore, we have

$$N_s = (p^n - 1)^{s-1} - N_{s-1}.$$

Consequently

$$N_s = (p^n - 1)^{s-1} - (p^n - 1)^{s-2} + \ldots + (-1)^s (p^n - 1)$$

$$= \frac{p^n - 1}{p^n} ((p^n - 1)^{s-1} + (-1)^s).$$

We also have, since the residue classes with respect to a rational prime modulus p form a finite field of order p, the

COROLLARY I. *If N is the number of incongruent solutions in the y's of the congruence, with p an odd prime,*

$$d_1 y_1^{a_1} + d_2 y^{a_2} + \ldots + d_s y_s^{a_s} \equiv 0 \pmod{p}$$

where $d_1 d_2 \ldots d_s \not\equiv 0 \pmod{p}$, $y_1 y_2 \ldots y_s \not\equiv 0 \pmod{p}$; $k_1, k_2, \ldots, k_s$, prime each to each with $(a_i, p - 1) = k_i$; $0 < k_i < p - 1, i = 1, 2, \ldots, s$ then

$$N = \frac{p - 1}{p} ((p - 1)^{s-1} + (-1)^s).$$

It may also be noted that the procedure used in the second proof of theorem II gives us a method for determining the solutions of (22).

[1] These PROCEEDINGS, **34,** 258–263 (1948), this will be referred to as E; **35,** 94–99 (1949), this will be referred to as C.

[2] Hasse, H., *Ber. Deut. Math. Verein.*, **35,** 1–55 (1926); **36,** 233–311 (1927); *Ergänzungsband*, **VI** (1930).

[3] Takagi, T., *J. Coll. Sci., Imp. Univ. Tokyo*, **42,** 16 (1920).

[4] Dickson, L. E., *Messenger Math.*, **38,** 14–33 (1908).

[5] These results are due to N. G. W. H. Beeger, unpublished.

[6] Vandiver, H. S., these PROCEEDINGS, **32,** 101–106 (1946).

[7] Cf. footnote 1 of E.

[8] The proof of this theorem we gave as a direct application of our general theorem I of C to the special case $s = 2$. Another proof may be obtained by using the result of Davenport and Hasse, *Jour. für Math.* (Crelle) 172, p. 174 relation (6.5) which gives the number of solutions of $ax^m + by^n = c$, $abc \neq 0$ in $F(p^n)$, where x and y are not restricted, as they were in our work, to non-zero values in $F(p^n)$. To do this we consider the possible solutions involving zero and subtract the number of them from the right-hand member of the relation (6.5) just referred to. We note that the number of solutions of $ax^m = c$ and $by^n = c$ are, respectively,

$$\sum_{\mu=0}^{m-1} \bar{x}^n \left(\frac{c}{a}\right); \sum_{\nu=0}^{n-1} \psi\left(\frac{c}{b}\right)$$

which when taken in connection with (6.5) yields a formula equivalent to our relation (18).

SUPERCONDUCTIVITY AND THE BOHR MAGNETON

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Each superconducting metal is characterized by a function $H_c(T)$ where H_c is a magnetic field in gauss and T is absolute temperature, which determines the boundary between those states in which the material shows superconducting properties and the region where the properties are normal. That is, for each T, the magnetic field H must be less than $H_c(T)$ for the metal to be in the superconducting state.

This functional relation plays an important rôle in the thermodynamics of superconductivity. If G_n is the Gibbs free energy of unit volume of the metal in its normal state and G_s is that in the superconducting state, but in zero magnetic field, then the Gibbs free energy of the superconductor in a magnetic field H is $G_s + H^2/8\pi$. At the field $H = H_c$ the normal and superconducting phases are in equilibrium so $G_n = G_s + H_c^2/8\pi$. This relation is the starting point of the Keesom-Rutgers-Gorter analysis of thermodynamic properties of superconductors, which has been so extraordinarily fruitful in the last decade.

The general form of $H_c(T)$ is usually that given by the formula

$$H_c(T) = a(T_0^2 - T^2)$$

which forms a basis for extrapolation of the observational data to $T = 0$. There is a nice graph of the data in figure 23 of Schoenberg's book on Superconductivity.[1] Aside from the question of the exact validity of this particular formula, the boundary curve has two particularly interesting parameters, namely, the critical field at absolute zero, $H_c(0)$, and the temperature T_0 for which H_c is zero.

If a long cylinder of superconducting material is magnetized by a field parallel to its axis it is known to behave as a perfect diamagnetic so that $B = 0$ inside it except for a skin layer of the order of 10^{-6} cm. thickness in which B drops from the external H down to zero in virtue of the building up of a $4\pi I$ oppositely directed to H and equal to it in magnitude when one gets below the skin depth. From this point of view, $I_c = H_c/4\pi$ is the maximum amount of magnetic moment per unit volume which can be set up in the superconductor without destruction of the superconducting state.

When one focusses attention on $I_c = H_c/4\pi$ as a limiting magnetic moment per unit volume it is natural to inquire whether empirically there is any relation between this number and the product $N\beta$ where N is the number of atoms per unit volume of the metal and β is the Bohr magneton.

No reason at all is offered for doing this except that these are of like dimensions and are both related to fundamental magnetic properties.

The results are shown in the table, the values of $H_c(0)$ having been read by rough extrapolation of the curves in Schoenberg's book and the values of N being calculated from standard density and atomic weight data.

TABLE 1

ELEMENT	$I_0 = N\beta$	$I_c(0) = H_c(0)/4\pi$	$K^{-1} = I_0/I_c$
Pb	306	63	4.8
Ta	513	92	5.6
Sn	271	26	10.4
Hg	376	35.8	10.5
In	356	23.1	15.4
Tl	324	15.9	20.4
Cb	506	207	2.5
Zr	393	17.9	21.9
Cd*	430	4	117
Al*	559	14	40

* The data for the last two, Cd and Al, are so uncertain that it is doubtful whether they should be included.

It seems remarkable that in all cases the quantities, I_0 and I_c, are of the same general order of magnitude. It is also striking that the values of I_0/I_c are roughly close to being an integral multiple of 5 (except columbium which requires a half quantum number!) so that the data here presented points roughly to a possible empirical relation,

$$I_c(0) = KN\beta, \tag{1}$$

in which, approximately, $K = {}^1/_{5n}$, where n is an integer (except that $n = {}^1/_2$ for columbium) characteristic of the metal. Whether this integer rule will stand up to a more critical study of the data remains to be seen.

It is interesting to compare the relation of I_c to $N\beta$ with the relation between mass transport of electrons by the superconducting electrons and Planck's angular momentum unit which was discovered by F. London.[2] London remarks that the electromagnetic measure of current per unit length of the cylinder wall, integrated through the skin depth is equal to $H/4\pi$ abamp./cm. where H is in gauss. To find the maximum mass transport in gram/cm. sec., $H_c/4\pi$ must therefore be multiplied by mc/e. London next remarks that this has the same dimensions as $n_e\hbar$ where $\hbar$ is the Planck unit of angular momentum and n_e is the number of superconducting electrons in unit volume, and then proceeds to present some evidence that these are of the same order of magnitude.

London's relation is thus

$$mcH/4\pi e = K'n_e\hbar$$

where K' is an empirical coefficient which he finds to be of the order of unity on the basis of some scanty data for lead and tin.

In view of the fact that the Bohr magneton, β, is related to the fundamental constants by $\beta = e\hbar/2mc$, London's relation can be written

$$I_c = H_c/4\pi = 2K'n_e\beta \qquad (2)$$

so that it is very closely similar to the relation given in equation (1). The two relations can be put together in such a way as to give an expression for the atomic concentration of superconducting electrons as follows:

$$n_e/N = K/2K'. \qquad (3)$$

Of course it is not really necessary to go through these relations to get a value of n_e/N. Basically, one finds n_e, in terms of London's theory, by making some kind of experimental determination of the skin-depth parameter, λ, from which n_e is calculated by the relation, $n_e = (mc^2/e^2)/4\pi\lambda^2$. The real difficulty lies in experimental measurement of λ. For this the available data are sparse and uncertain.

In London's paper[2] he only presents rough data for mercury and tin which lead him to say that $K' \sim 1$ for Hg and $K' \sim 1/6$ for Sn. Taking these values in combination with the values of K given in table 1, one gets

$$n_e = 1/20 \text{ for } Hg \text{ and } n_e/N = 1/3 \text{ for Sn.}$$

These are rather larger values than are usually supposed to be applicable, which may indicate that the experimental values for λ used by London are rather too small.

These considerations strongly point to the desirability of obtaining more accurate measurements of λ and of $H_c(0)$ for as many superconductors as possible.

[1] Schoenberg, D., *Superconductivity*, Cambridge University Press, 1938.

[2] London, F., *Rev. Mod. Phys.*, **17**, 310 (1945).

SEPARATION OF THE VARIABLES OF THE TWO-PARTICLE WAVE EQUATION

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The wave equation of two particles at points $P(x, y, z)$ and $\bar{P}(\bar{x}, \bar{y}, \bar{z})$, where x, y, z and $\bar{x}, \bar{y}, \bar{z}$ are rectangular-cartesian coördinates in euclidean 3-space is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2} + \frac{\partial^2}{\partial \bar{z}^2}\right)\psi + (E - V)\psi = 0,$$

where V is the potential. The distances r_1 and r_2 of the particles from the origin and their mutual distance from one another r_3 are given by

$$r_1^2 = x^2 + y^2 + z^2, \qquad r_2^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2, \qquad r_3^2 = \sum(x - \bar{x})^2. \quad (1)$$

Hylleraas[1] showed that for a solution $\psi(r_1, r_2, r_3)$ the above equation reduces to

$$\frac{\partial^2\psi}{\partial r_1^2} + \frac{\partial^2\psi}{\partial r_2^1} + 2\frac{\partial^2\psi}{\partial r_3^2} + \frac{r_1^2 - r_2^2 + r_3^2}{r_1 r_3}\frac{\partial^2\psi}{\partial r_1 \partial r_3} + \frac{r_2^2 - r_1^2 + r_3^2}{r_2 r_3}\frac{\partial^2\psi}{\partial r_2 \partial r_3} + \frac{2}{r_1}\frac{\partial\psi}{\partial r_1} + \frac{2}{r_2}\frac{\partial\psi}{\partial r_2} + \frac{4}{r_3}\frac{\partial\psi}{\partial r_3} + (E - V)\psi = 0. \quad (2)$$

If we put

$$x_1 = R_1, \qquad x_2 = R_2, \qquad x_3 = f(r_1, r_2, r_3), \quad (3)$$

where R_1 and R_2 are functions of r_1 and r_2 respectively, in the coördinates x_i the above equation is

$$R_1'^2\frac{\partial^2\psi}{\partial x_1^2} + R_2'^2\frac{\partial^2\psi}{\partial x_2^2} + \left[\left(\frac{\partial f}{\partial r_1}\right)^2 + \left(\frac{\partial f}{\partial r_2}\right)^2 + 2\left(\frac{\partial f}{\partial r_3}\right)^2 + \frac{r_1^2 - r_2^2 + r_3^2}{r_1 r_3}\frac{\partial f}{\partial r_1}\frac{\partial f}{\partial r_3} + \frac{r_2^2 - r_1^2 + r_3^2}{r_2 r_3}\frac{\partial f}{\partial r_2}\frac{\partial f}{\partial r_3}\right]\frac{\partial^2\psi}{\partial x_3^2} + 2R_1'\left(\frac{\partial f}{\partial r_1} + \frac{r_1^2 - r_2^2 + r_3^2}{2r_1 r_3}\frac{\partial f}{\partial r_3}\right)\frac{\partial^2\psi}{\partial x_1 \partial x_3} + 2R_2'\left(\frac{\partial f}{\partial r_2} + \frac{r_2^2 - r_1^2 + r_3^2}{r_2 r_3}\frac{\partial f}{\partial r_3}\right)\frac{\partial^2\psi}{\partial x_2 \partial x_3} + \left(R_1'' + \frac{2}{r_1}R_1'\right)\frac{\partial\psi}{\partial x_1} + \left(R_2'' + \frac{2}{r^2}R_2'\right)\frac{\partial\psi}{\partial x_2} + \left(\frac{\partial^2 f}{\partial r_1^2} + \frac{\partial^2 f}{\partial r_2^2} + 2\frac{\partial^2 f}{\partial r_3^2} + \frac{r_1^2 - r_2^2 + r_3^2}{r_1 r_3}\frac{\partial^2 f}{\partial r_1 \partial r_3} + \frac{r_2^2 - r_1^2 + r_3^2}{r_2 r_3}\frac{\partial^2 f}{\partial r_2 \partial r_3} + \frac{2}{r_1}\frac{\partial\psi}{\partial r_1} + \frac{2}{r_2}\frac{\partial f}{\partial r_2} + \frac{4}{r_3}\frac{\partial f}{\partial r_3}\right)\frac{\partial\psi}{\partial x_3} + (E - V)\psi = 0, \quad (4)$$

where primes denote derivatives.

In order that the coördinate system x_i be orthogonal f must satisfy the equations

$$\frac{\partial f}{\partial r_1} + \frac{r_1^2 - r_2^2 + r_3^2}{2r_1 r_3}\frac{\partial f}{\partial r_3} = 0, \qquad \frac{\partial f}{\partial r_2} + \frac{r_2^2 - r_1^2 + r_3^2}{2r_2 r_3}\frac{\partial f}{\partial_{r3}} = 0 \quad (5)$$

and there is no restriction upon R_1 and R_2. Eliminating $\partial f/\partial r_3$ from these equations we have

$$\frac{r_2^2 - r_1^2 + r_3^2}{r_2}\frac{\partial f}{\partial r_1} + \frac{r_2^2 - r_1^2 - r_3^2}{r_1}\frac{\partial f}{\partial r_2} = 0.$$

A solution of this equation is

$$f = \frac{r_3^2 - r_1^2 - r_2^2}{r_1 r_2} \tag{6}$$

and any function of this solution is also a solution. Equations (5) are satisfied identically by this value of f. Substituting it in equation (4) we obtain

$$R_1'^2\frac{\partial^2\psi}{\partial x_1^2} + R_2'^2\frac{\partial^2\psi}{\partial x_2^2} + \frac{r_1^2 + r_2^2}{r_1^2 r_2^2}[2(r_1^2 + r_2^2)r_3^2 - (r_1^2 - r_2^2)^2 - r_3^4] \times \frac{\partial^2\psi}{\partial x_3^2} + \left(R_1'' + \frac{2}{r_1}R_1'\right)\frac{\partial\psi}{\partial x_1} + \left(R_2'' + \frac{2}{r_2}R_2'\right)\frac{\partial\psi}{\partial x_2} + \frac{2}{r_1 r_2} \times \left(\frac{r_2^2 - r_3^2}{r_1^2} + \frac{r_1^2 - r_3^2}{r_2^2} + 2\right)\frac{\partial\psi}{\partial x_3} + (E - V)\psi = 0.$$

By means of

$$x_3 = \frac{r_3^2 - r_1^2 - r_2^2}{r_1 r_2} \tag{7}$$

this equation may be written

$$R_1'^2 \times \left[\frac{\partial^2\psi}{\partial x_1^2} + \left(\frac{R_1''}{R_1'^2} + \frac{2}{r_1}\frac{1}{R_1'}\right)\frac{\partial\psi}{\partial x_1}\right] + R_2'^2\left[\frac{\partial^2\psi}{\partial x_2^2} + \left(\frac{R_2''}{R_2'^2} + \frac{2}{r_2}\frac{1}{R_2'}\right)\frac{\partial\psi}{\partial x_2}\right] + \frac{r_1^2 + r_2^2}{r_1^2 r_2^2}(4 - x_3^2)\left(\frac{\partial^2\psi}{\partial x_3^2} + \frac{2x_3}{x_3^2 - 4}\frac{\partial\psi}{\partial x_3}\right) + (E - V)\psi = 0. \tag{8}$$

For a particular choice of R_1 and R_2 in (3) we write as the inverses of these equations

$$r_1 = h_1(x_1), \qquad r_2 = h_2(x_2). \tag{9}$$

If we put

$$V = R_1'^2 f_1(x_1) + R_2'^2 f_2(x_2) + \frac{r_1^2 + r_2^2}{r_1^2 r_2^2}(4 - x_3^2)f_3(x_3), \tag{10}$$

where f_i is an arbitrary function of x_i, and

$$\psi = X_1 X_2 X_3, \tag{11}$$

where X_i is a function of x_i alone, equation (8) may be written

$$R_1'^2\left[\frac{X_1''}{X_1}+\left(2\frac{h_1'}{h_1}-\frac{h_1''}{h_1'}\right)\frac{X_1'}{X_1}+\frac{E}{3}-f_1(x_1)\right]+$$
$$R_2'^2\left[\frac{X_2''}{X_2}+\left(2\frac{h_2'}{h_2}-\frac{h_2''}{h_2'}\right)\frac{X_2'}{X_2}+\frac{E}{3}-f_2(x_2)\right]+$$
$$\frac{r_1^2+r_2^2}{r_1^2r_2^2}(4-x_3^2)\left[\frac{X_3''}{X_3}+\frac{2x_3}{x_3^2-4}\frac{X_3'}{X_3}+\frac{E}{3}-f_3(x_3)\right]=0. \quad (12)$$

Hence the functions X_1, X_2, X_3 such that $\psi = X_1X_2X_3$ is a solution of equation (8) are solutions of the ordinary differential equations obtained by equating to zero the expressions in parentheses in (12).

From (1) and (7) it follows that, if θ is the angle between the vectors $0P$ and $0\overline{P}$,

$$x_3 = -2\cos\theta.$$

Consequently V as given in (10) may be written

$$V = F_1(r_1) + F_2(r_2) + 4\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}\right)\sin^2\theta f_3(-2\cos\theta), \quad (10a)$$

where F_1 and F_2 are

$$F_1 = R_1'^2 f_1(R_1), \qquad F_2 = R_2'^2 f_2(R_1)$$

and consequently can be given arbitrary values by suitable choice of R_1, R_2, $f_1(R_1)$, and $f_2(R_2)$.

Another orthogonal coördinate system x_i is given by

$$x_1 = \frac{r_1+r_2}{r_3}, \qquad x_2 = \frac{r_1-r_2}{r_3}, \qquad x_3 = \sqrt{r_1^2+r_2^2}, \quad (13)$$

as shown by Gronwall.[2] In this coördinate system equation (2) is

$$\frac{r_1^2+r_2^2}{r_1r_2r_3^2}\left[(x_1^2-1)\left(\frac{\partial^2\psi}{\partial x_1^2}+\frac{\partial}{\partial x_1}\log\frac{x_1^2-1}{x_1^2+x_2^2}\frac{\partial\psi}{\partial x_1}\right)+\right.$$
$$\left.(1-x_2^2)\left(\frac{\partial^2\psi}{\partial x_2^2}+\frac{\partial}{\partial x_2}\log\frac{1-x_2^2}{x_1^2+x_2^2}\frac{\partial\psi}{\partial x_2}\right)\right]+\frac{\partial^2\psi}{\partial x_3^2}+$$
$$\frac{5}{x_3}\frac{\partial\psi}{\partial x_3}+(E-V)\psi=0. \quad (14)$$

If we put

$$\psi = \sqrt{\frac{x_1^2+x_2^2}{(x_1^2-1)(1-x_2^2)}}\,\varphi,$$

the equation in φ is

$$\frac{r_1^2 + r_2^2}{r_1 r_2 r_3^3}\left[(x_1^2 - 1)\frac{\partial^2\varphi}{\partial x_1^2} + \frac{1}{x_1^2 - 1}\varphi + (1 - x_2^2)\frac{\partial^2\varphi}{\partial x_2^2} + \frac{1}{1 - x_2^2}\varphi\right] + \frac{\partial^2\varphi}{\partial x_3^2} + \frac{5}{x_3}\frac{\partial\varphi}{\partial x_3} + \left(E + \frac{2}{x_3^2} - V\right)\varphi = 0.$$

If then we put

$$V = \frac{r_1^2 + r_2^2}{r_1 r_2 r_3^2}\left[f_1\left(\frac{r_1 + r_2}{r_3}\right) + f_2\left(\frac{r_1 - r_2}{r_3}\right)\right] + f_3(r_1^2 + r_2^2), \tag{15}$$

where the functions f_i are arbitrary, a solution of (14) is given by

$$\psi = \sqrt{\frac{x_1^2 + x_2^2}{(x_1^2 - 1)(1 - x_2^2)}}\, X_1 X_2 X_3,$$

where

$$(x_1^2 - 1)X_1'' + \left(\frac{1}{x_1^2 - 1} - f_1(x_1)\right) X_1 = 0,$$

$$(1 - x_2^2)X_2'' + \left(\frac{1}{1 - x_2^2} - f_2(x_2)\right) X_2 = 0,$$

$$X_3'' + \frac{5}{x_3} X_3' + \left(E + \frac{3}{x_3} - f(x_3^2)\right) X_3 = 0.$$

[1] Hylleraas, E. A., Neue Berechnung der Energie des Heliums im grundzustande, sowie des tiefsten Terms von Ortho-Helium, *Z. Physik.*, **54**, 347–366 (1929).

[2] Gronwall, T. H., A special conformally euclidean space of three dimensions occurring in wave mechanics, *Ann. Math.*, Ser. 2, **34**, 279–293 (1932).

ERRATUM

In Table 2 of the article "Aggregation Phenomena in Polyvinyl Alcohol-Acetate Solutions" on page 367 of the July issue, 1949, the headings of columns two and four should be interchanged. F. F. NORD

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